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Asymptotic Properties of the Straight Line Estimator for a Renewal Function

(Sifat Asimptot bagi Penganggar Garis Lurus untuk Fungsi Pembaharuan)

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ABSTRACT

In estimation problems in renewal function, when the distribution is not known, nonparametric estimators of renewal function are used. Frees (1986a, Warranty analysis and renewal function estimation, Naval Res. Logist. Quart, 33, 361-372) proposed the nonparametric estimator of renewal function for large values of t. Frees's estimator is easy to apply in practice. It is a preferred estimator for large values of t. However, its statistical properties still have not been investigated in detailed. For this reason, in this study, we investigate asymptotic properties of this estimator such as consistency, asymptotic unbiasedness and asymptotic normality. Also Monte Carlo simulation study is given to assess the performance of this estimator according to value of renewal function. Simulation results indicate that in the large values of t, Frees estimator is sufficiently close to the renewal function for the Gamma distribution with various parameters.

Keywords: Asymptotic normality; asymptotic unbiasedness; consistency; nonparametric estimator; renewal function

ABSTRAK

Masalah anggaran dalam fungsi pembaharuan, apabila pengagihan tidak diketahui, penganggar tidak parametrik fungsi pembaharuan digunakan. Frees (1986a, analisis waranti dan anggaran fungsi pembaharuan, Naval Res. Logist. Quart, 33, 361-372) mencadangkan penganggar tidak parametrik fungsi pembaharuan bagi nilai besar t. Penganggar Frees adalah mudah untuk digunakan dalam amalan. Ia penganggar yang diutamakan bagi nilai besar t. Walau bagaimanapun, sifat statistiknya masih tidak dikaji dengan lebih mendalam. Untuk alasan ini, dalam kajian ini, kami mengkaji sifat asimptot penganggar ini seperti konsistensi, kesaksamaan asimptot dan kenormalan asimptot. Juga kajian simulasi Monte Carlo digunakan untuk menilai prestasi penganggar ini mengikut nilai fungsi pembaharuan. Hasil simulasi menunjukkan dalam nilai besar t, penganggar Frees hampir dengan fungsi pembaharuan agihan Gama dengan pelbagai parameter.

Kata kunci: Fungsi pembaharuan; kenormalan asimptot; ketaksamaan asimptot; konsisten; penganggar tak parametrik

INTRODUCTION

Renewal processes have a wide range of applications in probabilistic models of inventory theory, reliability theory, queuing theory and insurance applications. For example, in the analysis of most inventory processes it is customary to assume that the pattern of demands forms a renewal process. Most of the standard inventory policies induce renewal sequences, e.g. the times of replenishment of stock. In the analysis of most reliability processes, the renewal process is frequently used as a model for the reliability of a maintained system in which repair restores the system to as new condition and repair times are negligible in comparison to operating times. Reliability studies often interested in the number of failures of the unit over a given time interval.

The renewal function ($U(t)$) plays an important role in investigating of the renewal process (Feller 1971; Karlin & Taylor 1975; Ross 1996; Tijms 2003). Knowing the exact value of $U(t)$ or at least knowing the approximate value of $U(t)$ is helpful to solve a lot of interesting problems. To obtain $U(t)$, it is important to know the lifetime distribution, F, on the basis data collected from independent identically distribution. In many cases, the parametric form of the distribution F is not known. Thus, it is desirable to have a nonparametric estimator of the renewal function.

There are many studies on this topic in the literature. Vardi (1982) obtained an algorithm which produced a nonparametric maximum likelihood estimation of renewal function based on data. Frees (1986a) proposed estimation of a straight line approximation of the renewal function instead of direct estimation of the renewal function. This approximation is depended on a limit expression for large values of t. This estimator, $\hat{U}_n(t)$, is easy to apply in practice, especially for large values of t. Frees (1986a) noted that this estimator is not, in general, statistically consistent as the sample size n approaches infinity for small values of t. However, for large values of t, its statistical properties still have not been investigated in detailed. Frees (1986b) also proposed two nonparametric estimators based on the sum of the convolution with replacement and without replacement of the empirical distribution, respectively, especially for small values of t. But, to evaluate these estimators, it is needed to the considerable amount of computation, especially in the cases of large values of t. Unlike Frees's estimator $\hat{U}_n(t)$, this case is the disadvantage of these estimators.

At the same time, Schneider et al. (1990) proposed a nonparametric estimator for $U(t)$ and gave a comparison of this estimator with Frees's estimators. Grübel and Pitts (1993) introduced a nonparametric estimator based on empirical renewal function. Zhao and Rao (1997) estimated the renewal function by solving the renewal equation, incorporating a kernel estimate of the renewal density function. Guedon and Cocozza-Thivent (2003) addressed a problem of nonparametric estimation of renewal function from count data using Expectation Maximization algorithm. Markovich and Krieger (2006) gave a nonparametric estimator by using the bootstrap method. Also Bebbington et al. (2007) proposed estimator of the renewal function when the second moment is infinite. They developed confidence bands for the renewal and related functions.

In this study, we investigate statistical properties of Frees's estimator such as consistency, asymptotic unbiasedness and asymptotic normality. So this article is organized as follows. In the next section, the renewal process and renewal function are defined and Frees's estimator is presented. In the section that follows, we investigate its statistical properties. After that, Monte Carlo simulation study is given to assess the performance of this estimator according to value of $U(t)$ under Gamma distribution. Concluding remarks are summarized in the final section.

NONPARAMETRIC ESTIMATOR OF RENEWAL FUNCTION

In this section the nonparametric estimator of the renewal function given in Frees (1986a) is described. Let us give the definition of the renewal process and renewal function before we introduce the estimation.

Definition 2.1 Let $\{X_n, n=1, 2,\ldots\}$ be a sequence of independent and identically distributed positive valued random variables with distribution function F. Assume that F has mean μ and finite variance σ^2. Mathematical construction of the renewal process is defined as follows:

$$N(t)=\inf\{n\geq 1;\ T_n>t\} \text{ for } t\geq 0.$$

Here $T_n=\sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$ and $T_0=0$ are the arrival times or renewal sequence (Feller 1971).

Definition 2.2 The renewal function $U(t)$ is the expected number of renewals in an interval $(0, t)$. In other words, $U(t)$ is defined as follows:

$$\begin{aligned} U(t)=E\big(N(t)\big) &= \textstyle\sum_{n=1}^{\infty} nP\big(N(t)=n\big) \\ &= \textstyle\sum_{n=1}^{\infty} P\big(T_n\leq t\big)=\sum_{n=1}^{\infty} F^{n^*}(t), \end{aligned}$$

where $F^{n*}(t)$ denotes the n-fold convolution of F. When F is non-arithmetic distribution (i.e. the distribution function F is called non-aritmetic if the mass of F is not concentrated on a discrete set of points 0, λ, 2λ,... for some λ>0) and $E\left(X_i^2\right)<\infty$, $U(t)$ can be approximated for large values of t as $U(t) = U_a(t) + o(1)$.

Here $U_a(t) = t/\mu_1 + \mu_2/2\mu_1^2$ and μ_i (i=1, 2,..) is ith moment of the distribution F (Feller 1971), i.e., $\mu_1 = E\left(X_n^i\right)$ (i=1, 2,...). By using this expression, Frees (1986a) defined a nonparametric estimator of $U(t)$ given as:

$$\hat{U}_n(t)=\frac{t}{\hat{\mu}_1}+\frac{\hat{\mu}_2}{2\hat{\mu}_1^2},$$

where $\hat{\mu}_1=\overline{X}=\sum_{i=1}^{n} X_i/n$ and $\hat{\mu}_2=\overline{X}^2=\sum_{i=1}^{n} X_i^2/n$ are estimators of μ_1 and μ_2 based on the random sample $X_1, X_{2,\ldots,} X_n$. The estimator given in (2.1) is called Frees's estimator.

Remark Frees's estimator given in (2.1) is not, in general, statistically consistent as the sample size n approaches infinity for small values of t (Frees 1986a). However, for large values of t, its statistical properties still have not been investigated such as consistency, asymptotic unbiasedness and asymptotic normality. Aydoğdu (1997) investigated that $\hat{U}_n(t)$ estimator is an asymptotic unbiased estimator of $U(t)$ for only Gamma distribution and large values of t. In this study, statistical properties of this estimator in (2.1) are investigated for a large class including Gamma distribution when large values of t.

ASYMPTOTİC PROPERTIES OF FREES'S ESTIMATOR

In this section, we investigate some important statistical properties of Frees's estimator $\hat{U}_n(t)$ given in (2.1) for $U_a(t) = t/\mu_1 + \mu_2/2\mu_1^2$ for each fixed t. After that we show that $\hat{U}_n(t)$ converges $U(t)$ for large values of t. Let us first investigate the unbiasedness of this estimator. We can prove that $\hat{U}_n(t)$ estimator is an asymptotic unbiased estimator for fixed t.

Theorem 3.1 Suppose that $E\left(X_i^3\right)=\mu_3<\infty$, then

$$\lim_{n\to\infty} E\left(\hat{U}_n(t)\right)=\frac{t}{\mu_1}+\frac{\mu_2}{2\mu_1^2},$$

holds, that is, $\hat{U}_n(t)$ is asymptotic unbiased estimator of $U_a(t)$ for each fixed t.

Proof Let us first find the Taylor expansion of $\hat{U}_n(t)$ at μ_1 and μ_2

$$\begin{aligned} \hat{U}_n(t)=&\frac{t}{\mu_1}+\frac{\mu_2}{2_1^2}+\left(\overline{X}-\mu_1\right)\left(-\frac{t}{\mu_1^2}-\frac{\mu_2}{\mu_1^3}\right) \\ &+\left(\overline{X^2}-\mu_2\right)\frac{1}{2\mu_1^2}+\frac{1}{2}\left(\left(\overline{X}-\mu_1\right)^2\left(\frac{2t}{\mu_1^3}+\frac{3\mu_2}{\mu_1^4}\right)\right. \\ &\left.+2\left(\overline{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right)\left(-\frac{1}{\mu_1^3}\right)\right)+\hat{R}_2 \end{aligned}$$

where $\hat{R}_2$ is the remainder term. To obtain the expected value of $\hat{U}_n(t)$, firstly we need to prove that the expected value of $\hat{R}_2$ goes to zero as $n\to\infty$. It is provided in the Appendix A. Then, the expected value of remainder term $\hat{R}_2$ goes to zero as $n\to\infty$ and the expected value of $\hat{U}_n(t)$ could be given as shown:

$$E\left(\hat{U}_n(t)\right)=\frac{t}{\mu_1}+\frac{\mu_2}{2_1^2}+\left(-\frac{t}{\mu_1^2}-\frac{\mu_2}{\mu_1^3}\right)E\left(\left(\bar{X}-\mu_1\right)\right)$$
$$+\frac{1}{2\mu_1^2}E\left(\left(\overline{X^2}-\mu_2\right)\right)$$
$$+\frac{1}{2}\left(\left(\frac{2t}{\mu_1^3}+\frac{3\mu_2}{\mu_1^4}\right)E\left(\left(\bar{X}-\mu_1\right)^2\right)\right.$$
$$\left.+2\left(-\frac{1}{\mu_1^3}\right)\right)E\left(\left(\bar{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right)\right)\Bigg).$$

Here "≅" means asymptotically equivalency. It is seen that $E(\bar{X}-\mu_1)=0, E(\overline{X^2}-\mu_2)=0$, and $E((\bar{X}-\mu_1)^2)=\sigma_{\bar{X}}^2=\sigma^2/n$. Moreover, $E((\bar{X}-\mu_2)(\overline{X^2}-\mu_2))=Cov(\bar{X},\overline{X^2})=(\mu_3-\mu_1\mu_2)/n$ is derived from Appendix B. So is obtained as below:

$$E\left(\hat{U}_n(t)\right)\cong\frac{t}{\mu_1}+\frac{\mu_2}{2\mu_1^2}+\frac{1}{2n}\left\{\sigma^2\left(\frac{2t}{\mu_1^3}+\frac{3\mu_2}{\mu_1^4}\right)-\frac{2\left(\mu_3-\mu_1\mu_2\right)}{\mu_1^3}\right\}.$$

For fixed t, we have the bias of $\hat{U}_n(t)$ as follows:

$$Bias\left(\hat{U}_n(t)\right)\cong\frac{1}{2n}\left\{\sigma^2\left(\frac{2t}{\mu_1^3}+\frac{3\mu_2}{\mu_1^4}\right)-\frac{2\left(\mu_3-\mu_1\mu_2\right)}{\mu_1^3}\right\}.$$

Recall that $\mu_1>0, \mu_3<\infty$ and $t<\infty$, so $Bias\left(\hat{U}_n(t)\right)\rightarrow 0$ as $n\rightarrow\infty$. Hence,

$$E\left(\hat{U}_n(t)\right)\xrightarrow[n\rightarrow\infty]{}\frac{t}{\mu_1}+\frac{\mu_2}{2\mu_1^2},$$

holds. That is, it is seen that $\hat{U}_n(t)$ is asymptotic unbiased estimator of $U_a(t)$ for each fixed t.

Let us show this result on an example, when $X_1, X_2,\ldots, X_n$ are a random sample from Gamma distribution.

Example 3.1 Consider that $X_1, X_2,\ldots, X_n$ be independent and identically distributed random variables with Gamma distribution with parameter $\alpha>0$, $\beta>0$, i.e. probability density function (pdf) is $f(x;\alpha,\beta)=x^{\alpha-1}\exp(-x/\beta)/\Gamma(\alpha)\beta^{\alpha}$, $x>0$, First and second moments are as follows $\mu_1=\alpha\beta$ and $\mu_2=\alpha(\alpha+1)\beta^2$. So $U(t)$ and $\hat{U}_n(t)$ can be represented as follows:

$$U(t)=\frac{t}{\mu_1}+\frac{\mu_2}{2\mu_1^2}+o(1)=\frac{t}{\alpha\beta}+\frac{\alpha+1}{2\alpha}+o(1).$$

The expected value of the estimation $\hat{U}_n(t)=t/\bar{X}+\overline{X^2}/2(\bar{X})^2$ could be given as follows:

$$E\left(\hat{U}_n(t)\right)=E\left(\frac{t}{\bar{X}}\right)+E\left(\frac{\overline{X^2}}{2\left(\bar{X}\right)^2}\right)$$
$$=ntE\left(\frac{1}{\sum X_i}\right)+\frac{n}{2}E\left(\sum\left(\frac{X_i}{\sum X_i}\right)^2\right). \tag{3.5}$$

It is known that $Y=\sum_{i=1}^n X_i\sim Gamma(n\alpha,\ \beta)$ and $Z_k=X_i\big/\sum_{i=1}^n X_i\sim Beta(\alpha,(n-1)\alpha)$ (Hoog et al. 2005).

pdf of the Beta distribution with (α,β) is $f(z;\alpha,\beta)=z^{\alpha-1}(1-z)^{\beta-1}\Gamma(\alpha+\beta)/\Gamma(\alpha)\Gamma(\beta)$; $\alpha,\beta>0, 0<z<1$. So Eq. (3.5) can be rewritten as follows:

$$E\left(\hat{U}_n(t)\right)=ntE(1/Y)+n/2\,E\left(\sum_{i=1}^n Z_i^2\right). \tag{3.6}$$

We need to find $E(1/Y)$ and $\sum_{i=1}^n E\left(Z_i^2\right)$ to obtain $E\left(\hat{U}_n(t)\right)$ given in (3.6). So,

$$E(1/Y)=\int_0^\infty y^{-1}f_Y(y)dy=\int_0^\infty\frac{y^{n\alpha-2}e^{-y/\beta}}{\Gamma(n\alpha)\beta^{n\alpha}}dy=\frac{1}{\beta(n\alpha-1)},$$

$$E\left(Z_k^2\right)=\int_0^1 z^2 f_z(z)dz=\int_0^1\frac{z^{\alpha+1}(1-z)^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$
$$dz=\frac{\alpha+1}{n(n\alpha+1)},\quad k=1,\ldots,n.$$

Here $f_Y(y)$ and $f_Z(z)$ are the pdf of random variables Y and Z_k, respectively. Therefore, the $E\left(\hat{U}_n(t)\right)$ is given as follows:

$$E\left(\hat{U}_n(t)\right)=\frac{nt}{\beta(n\alpha-1)}+\frac{n(\alpha+1)}{2(n\alpha+1)}.$$

For fixed t, we have the bias of $\hat{U}_n(t)$ which is as follows:

$$Bias\left(\hat{U}_n(t)\right)\cong\frac{nt}{\beta(n\alpha-1)}+\frac{n(\alpha+1)}{2(n\alpha+1)}-\left(\frac{t}{\alpha\beta}+\frac{\alpha+1}{2\alpha}\right)$$
$$=\frac{t}{\alpha\beta(n\alpha-1)}.$$

Therefore, $Bias\left(\hat{U}_n(t)\right)\rightarrow 0$ for fixed t as $n\rightarrow\infty$, that is, $\hat{U}_n(t)$ is an asymptotic unbiased estimator of $U_a(t)$.

We need to obtain the variance of $\hat{U}_n(t)$ before show that $\hat{U}_n(t)$ is a consistent for $U_a(t)$ for each fixed t. So Lemma 3.1 gives the variance of $\hat{U}_n(t)$.

Lemma 3.1 Suppose that Then, the variance of $\hat{U}_n(t)$ can be represented as follows as $n\rightarrow\infty$

$$Var\left(\hat{U}_n(t)\right)=\frac{A}{n}-\frac{B}{n^2}+o\left(\frac{1}{n^2}\right),$$

where

$$A=\frac{1}{4\mu_1^2}\left\{4t^2\sigma^2\mu_1^2+4t\left(\mu_3\mu_1^2-\mu_2\mu_1^3+2\sigma^2\mu_2\mu_1\right)\right.$$
$$\left.+\left(4\mu_2^2\sigma^2+\mu_2^2\mu_1^2\right)\right\}$$

$$B=\frac{1}{4\mu_1^4}\left\{4t^2\sigma^4\mu_1^2-16t\sigma^2\mu_1^2\left(\mu_3-\mu_2\mu_1\right)+\right.$$
$$\left(16\mu_3^2\mu_1^2-32\mu_3\mu_2\mu_1^3-24\sigma^2\mu_3\mu_2\mu_1\right.$$
$$\left.\left.+16\mu_2^2\mu_1^4+24\sigma^2\mu_2^2\mu_1^2+9\sigma^4\mu_2^2+12\sigma^4\mu_2\mu_1\right)\right\}.$$

Here μ_k and σ^2 are kth moment and variance of distribution F, respectively.

Proof It is known that

$$Var\left(\hat{U}_n(t)\right)=Var\left(\frac{t}{\bar{X}}+\frac{\overline{X^2}}{2\bar{X}^2}\right)=E\left(\frac{t}{\bar{X}}+\frac{\overline{X^2}}{2\bar{X}^2}\right)^2-\left[E\left(\frac{t}{\bar{X}}+\frac{\overline{X^2}}{2\bar{X}^2}\right)\right]^2. \qquad (3.7)$$

The first term in (3.7) is calculated by obtaining Taylor expansion of the each terms at μ_1 and μ_2 (see Appendix B for details) given as

$$E\left(\frac{t}{\bar{X}}+\frac{\overline{X^2}}{2(\bar{X})^2}\right)^2\cong\frac{t^2}{\mu_1^2}+\frac{t\mu_2}{\mu_1^3}+\frac{\mu_2^2}{4\mu_1^4}+\frac{3t^2\sigma^2}{n\mu_1^5}-\frac{3t}{\mu_1^4}\left(\frac{\mu_3-\mu_1\mu_2}{n}\right)+\left(\frac{5\mu_2^2}{2\mu_1^6}\right)\frac{\sigma^2}{n}+\left(\frac{-2\mu_2}{\mu_1^5}\right)\left(\frac{\mu_3-\mu_1\mu_2}{n}\right)+\left(\frac{1}{4\mu_1^4}\right)\left(\frac{\mu_4-\mu_2^2}{n}\right). \qquad (3.8)$$

The second term given in (3.7) is the square of the term given in (3.3). Hence, $\left[E\left(t/\bar{X}+\overline{X^2}/2(\bar{X})^2\right)\right]^2$ term is given as:

$$\left[E\left(\frac{t}{\bar{X}}+\frac{\overline{X^2}}{2(\bar{X})^2}\right)\right]^2\cong\frac{t^2}{\mu_1^2}+\frac{t\mu_2}{\mu_1^3}+\frac{\mu_2^2}{4\mu_1^4}+\frac{4t\mu_2}{n\mu_1^3}+\frac{2\mu_2^2+2t^2\sigma^2-4t\mu_3}{n\mu_1^4}+\frac{4t\mu_2\sigma^2-2\mu_2\mu_3}{n\mu_1^5}+\frac{3\mu_2^2\sigma^2}{2n\mu_1^6}+\frac{4\mu_2^2}{n^2\mu_1^4}+\frac{4t\mu_2\sigma^2-8\mu_2\mu_3}{n^2\mu_1^5}+\frac{t^2\sigma^4+4\mu_3^2-4t\mu_3\sigma^2+6\mu_2^2\sigma^2}{n^2\mu_1^6}+\frac{3t\mu_2\sigma^2-6\mu_2\mu_3\sigma^2}{n^2\mu_1^7}+\frac{9\mu_2^2\sigma^4}{4n^2\mu_1^8}. \qquad (3.9)$$

Consequently, the variance of $\hat{U}_n(t)$, by using the result shown in (3.8) and (3.9), is given by

$$Var\left(\hat{U}_n(t)\right)=\frac{A}{n}-\frac{B}{n^2}+o\left(\frac{1}{n^2}\right), \qquad (3.10)$$

where

$$A=\frac{1}{4\mu_1^2}\left\{4t^2\sigma^2\mu_1^2+4t\left(\mu_3\mu_1^2-\mu_2\mu_1^3+2\sigma^2\mu_2\mu_1\right)\left(4\mu_2^2\sigma^2+\mu_2^2\mu_1^2+\mu_4\mu_1^2\right)\right\}$$

$$B=\frac{1}{4\mu_1^4}\left\{4t^2\sigma^4\mu_1^2+16t\sigma^2\mu_1^2\left(\mu_3-\mu_2\mu_1\right)+\left(16\mu_3^2\mu_1^2-32\mu_3\mu_2\mu_1^3+24\sigma^2\mu_3\mu_2\mu_1+16\mu_2^2\mu_1^4+24\sigma^2\mu_2^2+9\sigma^4\mu_2\mu_1\right)\right\}.$$

This completes the proof.

Corollary. 3.2 It can be shown that $|B|<\infty$, when $\mu_4<\infty$. Therefore, for fixed t,

$$Var\left(\hat{U}_n(t)\right)\cong\frac{A}{n}, \text{ as } n\to\infty. \qquad (3.11)$$

Now we can give the consistent estimator of $U_a(t)$ as given in Lemma 3.2 and Theorem 3.2.

Lemma 3.2 Suppose that $E\left(X_1^4\right)=\mu_4<\infty$. Then, $\hat{U}_n(t)$ is a consistent estimator of $U_a(t)$ where $U_a(t) = t/\mu_1 + \mu_2/2\mu_1^2$.

Proof Recall that $E\left(\hat{U}_n(t)\right)\cong U_a(t)$. To show that $\hat{U}_n(t)$ is a consistent estimator of $U_a(t)$, we need to prove that $\lim_{n\to\infty}P\left(\left|\hat{U}_n(t)-U_a(t)\right|\ge\varepsilon\right)=0$. It is well known that from Chebyshev inequality. Thus, we need to show that $\lim_{n\to\infty}E\left(\hat{U}_n(t)-U_a(t)\right)^2=0$ It is seen that $Bias(\hat{U}_n(t)$ in (3.4) and $Var(\hat{U}_n(t)$ in (3.11), $\lim_{n\to\infty}E\left(\hat{U}_n(t)-U_a(t)\right)^2=\lim_{n\to\infty}\left(Var\left(\hat{U}_n(t)\right)+\left(Bias\left(\hat{U}_n(t)\right)\right)^2\right)=0$. That is, $\hat{U}_n(t)$ is a consistent estimator for $U_a(t)$ for large values of t.

Now we can show that $\hat{U}_n(t)$ converges U(t) for large values of t.

Theorem 3.2 If $\hat{U}_n(t)\xrightarrow{P}U_a(t)$ and $U_a(t)\xrightarrow{P}U(t)$, then $\hat{U}_n(t)\xrightarrow{P}U(t)$.

Proof It is known that

$$P\left(\left|\left(\hat{U}_n(t)-U_a(t)\right)+\left(U_a(t)-U(t)\right)\right|<\varepsilon\right)\ge P\left(\left|\left(\hat{U}_n(t)-U_a(t)\right)\right|<\varepsilon/2 \text{ and } \left|\left(U_a(t)-U(t)\right)\right|<\varepsilon/2\right).$$

We have $P\left(\left|\hat{U}_n(t)-U_a(t)\right|<\varepsilon\right)\to1$ from Lemma 3.2. It is possible to find a number of $t_0>0$ for $\forall t > t_0$ so that $\left|\left(U_a(t)-U(t)\right)\right|<\varepsilon$ and $P\left(\left|\left(U_a(t)-U(t)\right)\right|<\varepsilon\right)=1$, $t > t_0$. If E_n and F_n are two sequences of events, then $P(E_n)\xrightarrow[n\to\infty]{}1$, $P(F_n)\xrightarrow[n\to\infty]{}1$ implies $P(E_n\cap F_n)\xrightarrow[n\to\infty]{}1$ (Lemma 2.1.2 in Lehmann 1998).
Hence,

$$P\left(\left|\left(\hat{U}_n(t)-U(t)\right)\right|<\varepsilon/2 \text{ and } \left|\left(U_n(t)-U(t)\right)\right|<\varepsilon/2\right)\to1,$$

therefore $P\left(\left|\left(\hat{U}_n(t)-U(t)\right)\right|<\varepsilon\right)\to1$, that is, $\hat{U}_n(t)\xrightarrow{P}U(t)$ for large values of t.

Remark Since $\hat{U}_n(t)\xrightarrow{P}U(t)$, then $\hat{U}_n(t)$ is a consistent and asymptotic unbiased estimator of $U(t)$ for large values of t.

To show asymptotic normality of the estimator $\hat{U}_n(t)$, first we need to follow Lemma 3.3 and Lemma 3.4.

Lemma 3.3 (Univariate Delta Method, Casella & Berger 2002) Let Y_n be a sequence of random variables with mean μ and variance σ^2 that satisfies $\sqrt{n}(Y_n-\mu)\xrightarrow{d}N(0,\sigma^2)$. For a given function g(Y_n) and specific value of μ, supposed that g'(μ) exists and is not 0. Then,

$$\sqrt{n}\left(g(Y_n)-g(\mu)\right)\xrightarrow{d}N\left(0,\sigma^2\left(g'(\mu)\right)^2\right).$$

Lemma 3.4 (Multivariate Delta Method, Casella & Berger 2002) Define the random vector $\mathbf{Y}=(Y_1,\ldots,Y_r)$ with mean $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_r)$ and covariance $Cov(Y_i, Y_j) = \sigma_{ij}$. Let $\mathbf{Y}^{(1)},\ldots,\mathbf{Y}^{(n)}$ be a random samples of the population $\mathbf{Y}$ and $\hat{Y}_i=\sum_{k=1}^{n}Y_i^{(k)}$, i=1,...,r be the sum of observation for each variable. For a given function g($\mathbf{Y}$) with continues first partial derivatives and a specific value of $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_r)$ for which $\tau^2=\sum_i\sum_j\sigma_{ij}g_i'(\mu)g_j'(\mu)>0$. Then $\sqrt{n}\left(g\left(\hat{Y}^1,\ldots,\hat{Y}^{(n)}\right)-g(\mu_1,\ldots,\mu_p)\right)\xrightarrow{d}N(0,\tau^2)$. Here $g'_i(\mu) = \partial g(\mu_1,\ldots,\mu_r)/\partial\mu_i$ (i=1,…r). Now we can give asymptotic normality of the estimator $\hat{U}_n(t)$.

Theorem 3.3 Suppose that $E\left(X_i^4\right)=\mu_4<\infty$ then $\hat{U}_n(t)$ is asymptotically normal, that is, $\sqrt{n}\left(\hat{U}_n(t)-U(t)\right)\xrightarrow{d}N(0,\Delta^2)$ where $\Delta^2=\frac{t^2\sigma^2}{\mu_1^4}+\frac{1}{4n\mu_1^6}\left(\mu_4\mu_1^2-4\mu_3\mu_2\mu_1+4\mu_2^3-\mu_2^2\mu_1^2\right)$. Here μ_k and σ^2 are kth moment and variance of distribution F.

Proof Firstly, we investigate the asymptotic distribution of first term $t/\bar{X}$ in (2.1) from Lemma 3.3. Let $Y_n = \bar{X}$ be a sequence of random variables that satisfies According to Lemma 3.3, the given function is $g(Y_n) = t/Y_n$ and $g'(\mu_1)$ exists such that $g'(\mu_1) = -t/\mu_1^2$ for Y_n >0. Then $\sqrt{n}\left(t/\bar{X}-t/\mu_1\right)\xrightarrow{d}N\left(0,\sigma^2\frac{t^2}{\mu_1^4}\right)$.

Using the same way, we investigate the asymptotic distribution of second term $(\bar{X}^2/2(\bar{X})^2)$ in (2.1) from Lemma 3.4. Let $x = \bar{X}$ and $y = \bar{X}^2$ with mean $E(\bar{X}) = \mu_1$ and $E(\bar{X}^2) = \mu_2$ and covariance $Cov(\bar{X}, \bar{X}^2) = (\mu_2 - \mu_1\mu_2)/n$. The first partial derivatives of $g(x,y) = y/2x^2$ are $\partial g(x,y)/\partial x = -y/x^3$ and $\partial g(x,y)/\partial y = 1/2x^2$.

Then, from Lemma 3.4.

$$\sqrt{n}\left(\bar{X}^2/2\bar{X}^2-\mu_2/2\mu_1^2\right)\xrightarrow{d}N(0,\tau^2),$$

where

$$\tau^2=\frac{1}{n}\left(\frac{(\mu_2-\mu_1^2)\mu_2^2}{\mu_1^6}+\frac{\mu_4-\mu_2^2}{4\mu_1^4}-\frac{(\mu_3-\mu_1\mu_2)\mu_2}{\mu_1^5}\right)$$
$$=\frac{1}{4n\mu_1^6}\left(\mu_4\mu_1^2-4\mu_3\mu_2\mu_1+4\mu_2^3+-\mu_2^2\mu_1^2\right).$$

By applying Slutsky theorem (Lehmann 1998), we have that

$$\sqrt{n}\left(\hat{U}(t)-U(t)\right)\xrightarrow{d}N(0,\Delta^2).$$

where

$$\Delta^2=\frac{t^2}{\mu_1^4}\sigma^2+\frac{1}{4n\mu_1^6}\left(\mu_4\mu_1^2-4\mu_3\mu_2\mu_1+4\mu_2^3+-\mu_2^2\mu_1^2\right)$$

This completes the proof.

As a result, it is shown that $\hat{U}_n(t)$ is consistent, asymptotic unbiased and asymptotically normal estimator of $U(t)$ for large values of t.

SIMULATION

In this section, a Monte Carlo simulation study is given to assess the performance of the estimator $\hat{U}_n(t)$ according to value of $U(t)$. We investigate the effect of the different values of n and t. The following values $t \in$ (5, 15, 25, 50) are used for $n \in$ (5, 10, 15, 20, 30, 50, 100). We generate samples with the known pdf of a Gamma distribution ($f(x,\alpha,\beta) = x^{\alpha-1}\exp(-x/\beta)/\Gamma(\alpha)\beta^{\alpha}$) with the parameters (1, 1), (2, 1) and (3, 1). In Tables 1-3 one can see the performance of the estimator $\hat{U}_n(t)$ calculated by 10000 repeated samples from the given distribution by using MATLAB program. $\tilde{E}\left(\hat{U}_n(t)\right)$ denotes Monte Carlo estimation of expected value of $\hat{U}_n(t)$. In addition, $\delta=\left(\left|U(t)-\tilde{E}\left(\hat{U}_n(t)\right)\right|/U(t)\right)\times 100\%$ and AP=1- d denote the relative error and accuracy percentage between the $U(t)$ and $\tilde{E}\left(\hat{U}_n(t)\right)$, respectively.

Here $U(t) = t + 1$.

Here $U(t) = t/2 + 3/4 + (1/4)\exp(-2t)$.

Here $U(t) = t/3 + 2/3 + (2\sqrt{3}/9)\exp(-3t/2)\cos(\sqrt{3}t/2 - \pi/6)$.

In Table 1, APs between the $U(t)$ and $\tilde{E}\left(\hat{U}_n(t)\right)$ increase quickly as n increases for each t under the Gamma (1, 1) distribution. For example, in all cases, APs are greater than 90% when n=15. It is also seen that AP is negatively affected by the large value of t for small n. For example, while AP is 81.82% for n=5 and t=5, AP is 75.65% for n=5 and t=50. However, this case improves as n increases.

As seen from Table 2, APs increase as n increases for each t. It is observed that the APs between the $U(t)$ and $\tilde{E}\left(\hat{U}_n(t)\right)$ are positively affected by that α parameter of Gamma distribution increases. For example, for n=5 and t=50, AP is 89.582% in Table 2 while it is 76.53% in Table 1. Moreover, in Table 3 it is observed that this result is similarly obtained. For example, for n=5 and t=50, AP is 93.307% in Table 3 while it is 89.582% in Table 2.

CONCLUSION

Estimation problems in renewal function have been studied by many authors. When the shape of the distribution cannot

TABLE 1. A comparison of the values of $\tilde{E}\left(\hat{U}_n(t)\right)$ and $U(t)$ for Gamma (1,1) distribution

t	$U(t)$	n	$\tilde{E}\left(\hat{U}_n(t)\right)$	δ(%)	AP	t	$U(t)$	n	$\tilde{E}\left(\hat{U}_n(t)\right)$	δ(%)	AP
		5	7,09	18,18	81,82			5	32,10	23,47	76,53
		10	6,47	7,88	92,12			10	28,71	10,44	89,56
		15	6,30	5,02	94,98			15	27,75	6,72	93,28
5	6,00	20	6,22	3,61	96,39	25	26,00	20	27,28	4,92	95,08
		30	6,14	2,37	97,63			30	26,81	3,11	96,89
		50	6,08	1,29	98,71			50	26,49	1,88	98,12
		100	6,04	0,71	99,29			100	26,24	0,92	99,08
		5	19,55	22,20	77,80			5	63,42	24,35	75,65
		10	17,59	9,92	90,08			10	56,41	10,61	89,39
		15	17,00	6,22	93,78			15	54,52	6,90	93,10
15	16,00	20	16,75	4,67	95,33	50	51,00	20	53,61	5,11	94,89
		30	16,51	3,21	96,79			30	52,62	3,17	96,83
		50	16,29	1,82	98,18			50	51,97	1,91	98,09
		100	16,13	0,84	99,16			100	51,47	0,92	99,08

TABLE 2. A comparison of the values of $\tilde{E}\left(\hat{U}_n(t)\right)$ and $U(t)$ for Gamma (2,1) distribution

t	$U(t)$	n	$\tilde{E}\left(\hat{U}_n(t)\right)$	δ(%)	AP	t	$U(t)$	n	$\tilde{E}\left(\hat{U}_n(t)\right)$	δ(%)	AP
		5	3,461	6,480	93,520			5	14,546	9,782	90,218
		10	3,350	3,078	96,922			10	13,869	4,672	95,328
		15	3,311	1,880	98,120			15	13,655	3,056	96,944
5	3,250011	20	3,295	1,399	98,601	25	13,25	20	13,551	2,275	97,725
		30	3,279	0,897	99,103			30	13,443	1,456	98,544
		50	3,268	0,557	99,443			50	13,366	0,872	99,128
		100	3,259	0,266	99,734			100	13,306	0,423	99,577
		5	9,026	9,401	90,599			5	28,433	10,418	89,582
		10	8,597	4,203	95,797			10	27,026	4,954	95,046
		15	8,481	2,800	97,200			15	26,584	3,238	96,762
15	8,25	20	8,425	2,117	97,883	50	25,75	20	26,374	2,425	97,575
		30	8,369	1,442	98,558			30	26,154	1,568	98,432
		50	8,316	0,800	99,200			50	26,005	0,991	99,009
		100	8,286	0,430	99,570			100	25,875	0,487	99,513

TABLE 3. A comparison of the values of $\tilde{E}\left(\hat{U}_n(t)\right)$ and $U(t)$ for Gamma (3,1) distribution

t	$U(t)$	n	$\tilde{E}\left(\hat{U}_n(t)\right)$	δ(%)	AP	t	$U(t)$	n	$\tilde{E}\left(\hat{U}_n(t)\right)$	δ(%)	AP
		5	2,411	3,328	96,672			5	9,554	6,159	93,841
		10	2,370	1,568	98,432			10	9,273	3,030	96,970
		15	2,357	1,030	98,970			15	9,176	1,959	98,041
5	2,333166	20	2,350	0,737	99,263	25	9,00	20	9,131	1,460	98,540
		30	2,345	0,512	99,488			30	9,087	0,967	99,033
		50	2,340	0,305	99,695			50	9,049	0,540	99,460
		100	2,337	0,149	99,851			100	9,024	0,268	99,732
		5	5,982	5,561	94,439			5	18,494	6,693	93,307
		10	5,821	2,729	97,271			10	17,891	3,216	96,784
		15	5,767	1,766	98,234			15	17,693	2,077	97,923
15	5,666667	20	5,741	1,305	98,695	50	17,33333	20	17,598	1,526	98,474
		30	5,715	0,861	99,139			30	17,502	0,975	99,025
		50	5,695	0,503	99,497			50	17,444	0,639	99,361
		100	5,682	0,269	99,731			100	17,392	0,336	99,664

be assumed to be known, nonparametric estimators of renewal function are used. However, most of them are not easy to apply in practice, especially for large values of t. Because of the simplicity of Frees's estimator $\hat{U}_n(t)$, it can be used easily in practice. Consequently, $\hat{U}_n(t)$ is the preferred estimator for large value of t. So in present work, we investigated statistical properties of Frees's estimator $\hat{U}_n(t)$. In fact, it can be important to prove asymptotic properties of this estimator when use in stochastic models including the complicated function associated with renewal function, for example inventory model of type (s, S). This model has been extensively considered in recent years (Khaniyev & Atalay 2010; Khaniev & Mammadova 2006; Khaniyev et al. 2013). We proved the asymptotic properties of Frees's estimator $\hat{U}_n(t)$ such as consistency, asymptotic unbiasedness and asymptotic normality. In simulation study, it is observed that according to values of t and n, $\tilde{E}\left(\hat{U}_n(t)\right)$ are sufficiently close to the $U(t)$ for Gamma distribution with various parameters.

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APPENDIX A

The expected value of the remainder term $\hat{R}_2$ goes to zero as $n\to\infty$. To show this, initially, we need to obtain Lagrange form of the remainder term $\hat{R}_2$ as follows:

$$\hat{R}_2 = \hat{R}_2\left(\bar{X},\overline{X^2}\right) = \frac{-4\left(\bar{X}-\mu_1\right)^2\left(\mu_2+\theta\left(\overline{X^2}-\mu_2\right)\right)}{2\left(\mu_1+\theta\left(\bar{X}-\mu_1\right)\right)^5} + \frac{-2t\left(X-\mu_1\right)^3}{2\left(\mu_1+\theta\left(\bar{X}-\mu_1\right)\right)} + \frac{\left(\bar{X}-\mu_1\right)^2\left(\overline{X^2}-\mu_2\right)}{2\left(\mu_1+\theta\left(\bar{X}-\mu_1\right)\right)^4},$$

where 0<θ<1. According to the law of large numbers, $\bar{X}\xrightarrow{P}\mu_1$ and $\overline{X^2}\xrightarrow{P}\mu_2$, then n can be chosen so large that $\left|\bar{X}-\mu_1\right|<\frac{\delta_1}{2}$ and $\left|\overline{X^2}-\mu_2\right|<\frac{\delta_2}{2}$ where $0<\theta_1<1$ and $0<\theta_2<1$. We have inequality as shown below:

$$\mu_1+\theta\left(\bar{X}-\mu_1\right)\geq\mu_1-\frac{\theta\delta_1}{2} \text{ and } \mu_2+\theta\left(\overline{X^2}-\mu_2\right)\geq\mu_2-\frac{\theta\delta_2}{2}.$$

So we obtain the upper bound of $|\hat{R}_2|$ as follows:

$$\left|\hat{R}_2\left(\bar{X},\overline{X^2}\right)\right|\leq\frac{4\left(\delta_1/2\right)^2\left(\mu_2+\left(\theta\delta_2/2\right)\right)}{2\left|\mu_1-\left(\theta\delta_1/2\right)\right|^5}+\frac{t\left(\delta_1/2\right)^3}{\left|\mu_1-\left(\theta\delta_1/2\right)\right|^4} +\frac{\left(\delta_1/2\right)^2\left(\delta_2/2\right)}{2\left|\mu_1-\left(\theta\delta_1/2\right)\right|^4}.$$

δ_1 $(0<\delta_1<1)$ is chosen so small that let $|\mu_2-(\theta\delta_1/2)|\geq\mu_1/2$. Therefore,

$$\left|\hat{R}_2\left(\bar{X},\overline{X^2}\right)\right|\leq\frac{4\left(\delta_1/2\right)^2\left(\mu_2+\left(\theta\delta_2/2\right)\right)}{2\left(\mu_1/2\right)^5}+t\frac{\left(\delta_1/2\right)^3}{\left(\mu_1/2\right)^4}+\frac{\left(\delta_1/2\right)^2\left(\delta_2/2\right)}{2\left(\mu_1/2\right)^4}$$
$$=\frac{168_1^2\left(\mu_2+\left(\theta\delta_2/2\right)\right)}{\mu_1^5}+t\frac{2\delta_1^3}{\mu_1^4}+\frac{\delta_1^2\delta_2}{\mu_1^4}, \text{ with probability 1.}$$

δ_2 $(0<\delta_2<1)$ is chosen so small that let $\theta\delta_2/2\leq\mu_2$. Therefore,

$$\left|\hat{R}_2\left(\bar{X},\overline{X^2}\right)\right|\leq\frac{32\mu_2}{\mu_1^5}\delta_2^2+\frac{2t}{\mu_1^4}\delta_1^3+\frac{1}{\mu_1^4}\delta_1^2\delta_2, \quad \text{with probability 1.}$$

Let $\max(\delta_1;\delta_2)=\delta$, in this case

$$\left|\hat{R}_2\left(\bar{X},\overline{X^2}\right)\right|\leq\frac{32\mu_2}{\mu_1^5}\delta_2^2+\frac{2t}{\mu_1^4}\delta_1^3+\frac{1}{\mu_1^4}\delta^3, \quad \text{with probability 1.}$$

and with respect to $\delta^3<\delta^2$,

$$\left|\hat{R}_2\left(\bar{X},\overline{X^2}\right)\right|\leq\frac{32\mu_2+2t\mu_1+\mu_1}{\mu_1^5}\delta^2, \quad \text{with probability 1.}$$

Recall that $0<\mu_1<\infty$ and $0<\mu_2<\infty$, so it is seen that $(32\mu_2+2t\mu_1+\mu_1)/\mu_1^5=c<\infty$. For each $\delta\in(0,1)$, n can be chosen so large that $\left|\hat{R}_2\left(\bar{X},\overline{X^2}\right)\right.$ with probability 1. It is known that if $X_1\leq X_2$ in probability 1, then $E(X_1)\leq E(X_2)$ (Roussas 1997). So, for each $\delta\in(0,1)$,

$$\left|E\left(\hat{R}_2\left(\bar{X},\overline{X^2}\right)\right)\right|\leq E\left(\left|\hat{R}_2\left(\bar{X},\overline{X^2}\right)\right|\right)\leq E\left(c\delta^2\right)=c\delta^2.$$

Therefore, for each $\delta\in(0,1)$, $E\left|\hat{R}_2\left(\bar{X},\overline{X^2}\right)\right.\to0$ as $n\to\infty$.

APPENDIX B

To show unbiasedness properties of this estimator, we first need to prove following Proposition B.1 about the covariance between $\bar{X}$ and $\overline{X^2}$.

Proposition B.1. Let $X_1, X_{2,\ldots,} X_n$ be independent and identically distributed random variables with distribution function F and μ_i (i=1, 2,...) are ith moment of the distribution. $\overline{X^k} = \sum_{i=1}^{n} X_i^k / n$ (k=1, 2,...) and $\overline{X^m} = \sum_{i=1}^{n} X_i^m / n$ (m=1, 2,…) are kth and mth sample moment, respectively. So the covariance between $\overline{X^k}$ and $\overline{X^m}$ can be given as

$$Cov\left(\overline{X^k}\,\overline{X^m}\right) = \left(\mu_{k+m} - \mu_k \mu_m\right)/n.$$

Proof The covariance between $\overline{X^k}$ and $\overline{X^m}$ is obtained as follows

$$Cov\left(\overline{X^k}\,\overline{X^m}\right) = E\left[\left(\overline{X^k} - \mu_k\right)\left(\overline{X^m} - \mu_m\right)\right] = E\left(\overline{X^k}\,\overline{X^m}\right) - \mu_k \mu_m. \tag{B.1}$$

Here $E\left(\overline{X^k}\,\overline{X^m}\right)$ term is obtained as

$$E\left(\overline{X^k}\,\overline{X^m}\right) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^k \frac{1}{n}\sum_{i=1}^{n} X_i^m\right) = \frac{1}{n^2} E\left(\sum_{i=1}^{n} X_i^{k+m} + 2\sum_{1 \le i < j \le n} X_i^k X_j^m\right)$$
$$= \frac{\mu_{k+m}}{n} + \mu_k \mu_m - \frac{\mu_k \mu_m}{n}$$

So (B.1) could be written as

$$Cov\left(\overline{X^k}\,\overline{X^m}\right) = \frac{\mu_{k+m}}{n} + \mu_k \mu_m - \frac{\mu_k \mu_m}{n} - \mu_k \mu_m = \frac{\mu_{k+m} - \mu_k \mu_m}{n}.$$

This completes the proof.

Corollary B.1 The covariance between $\bar{X}$ and $\overline{X^2}$ can be given as

$$Cov\left(\overline{X^k}\,\overline{X^m}\right) = (\mu_3 - \mu_1 \mu_2)/n.$$

Remark As seen from Corollary B.1 $\lim_{n\to\infty} Cov\left(\bar{X}\,\overline{X^2}\right) = 0$ when $\mu_3 < \infty$. This result showed that $\bar{X}$ and $\overline{X^2}$ are asymptotically uncorrelated.

$E\left(t/\bar{X}+\overline{X^2}/2\bar{X}^2\right)^2$ term is rewritten as follows:-

$$E\left(\frac{t}{\bar{X}}+\frac{\overline{X^2}}{2\bar{X}^2}\right)=t^2E\left(\frac{1}{\bar{X}^2}\right)+tE\left(\frac{\overline{X^2}}{\bar{X}^2}\right)+\frac{1}{4}E\left(\frac{\left(\overline{X^2}\right)^2}{\bar{X}^4}\right). \tag{C.1}$$

Initially, we need to obtain the Taylor expansion of each term at μ_1 and μ_2 in (C.1). For first term in (C.1), we have the following result:

$$\frac{1}{\bar{X}^2}=\frac{1}{\mu_1^2}+\left(-\frac{2}{\mu_1^3}\right)\left(\bar{X}-\mu_1\right)+\frac{3}{\mu_1^4}\left(\bar{X}-\mu_1\right)^2+\hat{R}_2.$$

According to the law of large numbers, $\bar{X}\xrightarrow{P}\mu_1$ and $\overline{X^2}\xrightarrow{P}\mu_2$, by using the same way given in Appendix A, it can be shown that the expected value of the remainder term goes to zero as $n\to\infty$. Therefore,

$$E\left(\frac{1}{\bar{X}^2}\right)\cong\frac{1}{\mu_1^2}+\left(-\frac{2}{\mu_1^3}\right)E\left(\bar{X}-\mu_1\right)+\frac{3}{\mu_1^4}E\left(\bar{X}-\mu_1\right)^2.$$

Here $E\left(\bar{X}-\mu_1\right)=0$ and $E\left(\left(\bar{X}-\mu_1\right)^2\right)=\sigma_{\bar{X}}^2=\sigma^2/n$. Hence

$$E\left(\frac{1}{\left(\bar{X}\right)^2}\right)\cong\frac{1}{\mu_1^2}+\frac{3\sigma^2}{n\mu_1^4}. \tag{C.2}$$

For second term in (C.1), we have the following result

$$\begin{aligned}\frac{\overline{X^2}}{\bar{X}^3}&=\frac{\mu_2}{\mu_1^3}+\left(-\frac{3\mu_2}{\mu_1^4}\right)\left(\bar{X}-\mu_1\right)+\frac{1}{\mu_1^3}\left(\overline{X^2}-\mu_2\right)\\&+\frac{1}{2}\left[\frac{12\mu_2}{\mu_1^5}\left(\bar{X}-\mu_1\right)^2+2\left(-\frac{3}{\mu_1^4}\left(\bar{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right)\right)\right]+\hat{R}_2.\end{aligned}$$

According to law of large numbers, $\bar{X}\xrightarrow{P}\mu_1$ and $\overline{X^2}\xrightarrow{P}\mu_2$, therefore,

$$\begin{aligned}E\left(\frac{\overline{X^2}}{\bar{X}^3}\right)&\cong\frac{\mu_2}{\mu_1^3}+\left(-\frac{3\mu_2}{\mu_1^4}\right)E\left(\bar{X}-\mu_1\right)+\frac{1}{\mu_1^3}E\left(\overline{X^2}-\mu_2\right)\\&+\frac{1}{2}\left[\frac{12\mu_2}{\mu_1^5}E\left(\bar{X}-\mu_1\right)^2+2\left(-\frac{3}{\mu_1^4}E\left(\bar{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right)\right)\right].\end{aligned}$$

Here $E\left(\overline{X^2}-\mu_2\right)=0$ and from Corollary B.1 we have $E\left(\left(\bar{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right)\right)=\left(\mu_3-\mu_1\mu_2\right)/n$. Then,

$$E\left(\frac{\left(\overline{X^2}\right)}{\bar{X}^4}\right)\cong\frac{t\mu_2}{\mu_1^3}+\frac{6t\mu_2\sigma^2}{n\mu_1^5}-\frac{3t}{\mu_1^4}\left(\frac{\mu_3-\mu_1\mu_2}{n}\right). \tag{C.3}$$

For third term in (C.1), we have the following result

$$\begin{aligned}\frac{\left(\overline{X^2}\right)}{\bar{X}^4}&=\frac{\mu_2^2}{\mu_1^4}+\left(-\frac{4\mu_2^2}{\mu_1^5}\right)\left(\bar{X}-\mu_1\right)+2\left(\frac{\mu_2}{\mu_1^4}\right)\left(\overline{X^2}-\mu_2\right)\\&+\frac{1}{2}\left[\frac{20\mu_2^2}{\mu_1^6}\left(\bar{X}-\mu_1\right)^2+2\left(\frac{-8\mu_2}{\mu_1^5}\right)\left(\bar{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right)+\left(\frac{2}{\mu_1^4}\right)\left(\overline{X^2}-\mu_2\right)^2\right]+\hat{R}_2.\end{aligned}$$

According to the law of large numbers, $\bar{X} \xrightarrow{P} \mu_1$ and $\overline{X^2} \xrightarrow{P} \mu_2$, therefore,

$$E\left(\frac{\left(\overline{X^2}\right)}{\bar{X}^4}\right) \cong \frac{\mu_2^2}{\mu_1^4} + \left(-\frac{4\mu_2^2}{\mu_1^5}\right) E\left(\bar{X}-\mu_1\right) + 2\left(\frac{\mu_2}{\mu_1^4}\right) E\left(\overline{X^2}-\mu_2\right)$$
$$+\frac{1}{2}\left[\frac{20\mu_2^2}{\mu_1^6} E\left(\bar{X}-\mu_1\right)^2 + 2\left(\frac{-8\mu_2}{\mu_1^5}\right) E\left(\bar{X}-\mu_1\right)\left(\overline{X^2}-\mu_2\right) + \left(\frac{2}{\mu_1^4}\right) E\left(\overline{X^2}-\mu_2\right)^2\right]. \tag{C.4}$$

Here $E\left(\overline{X^2}-\mu_2\right)^2$ term is found as follows:-

$$E\left(\overline{X^2}-\mu_2\right)^2 = Var\left(\overline{X^2}\right) = E\left(\overline{X^2}\right)^2 - \left(E\left(\overline{X^2}\right)\right)^2.$$

$$E\left(\overline{X^2}\right)^2 = E\left(\left(\left(X_1^2+X_2^2+\ldots+X_n^2\right)/n\right)^2\right) = \frac{1}{n^2} E\left(\sum_{i=1}^{n} X_i^4 + 2\sum_{i<j}\sum X_i^2 X_j^2\right),$$

$$E\left(\overline{X^2}\right)^2 = \frac{1}{n^2}\left(n\mu_4 + n(n-1)\mu_2^2\right) = \mu_2^2 + \frac{\mu_4-\mu_2^2}{n}.$$

In this case,

$$E\left(\overline{X^2}-\mu_2\right)^2 = \mu_2^2 + \frac{\mu_4-\mu_2^2}{n} - \mu_2^2 = \frac{\mu_4-\mu_2^2}{n}. \tag{C.5}$$

By using the result given in (C. 5), we can rewrite (C. 4) as follows:

$$E\left(\frac{\left(\overline{X^2}\right)^2}{\left(\bar{X}\right)^4}\right) \cong \frac{\mu_2^2}{\mu_1^4} + \left(\frac{10\mu_2^2}{2\mu_1^6}\right)\frac{\sigma^2}{n} + \left(\frac{-8\mu_2}{\mu_1^5}\right)\left(\frac{\mu_3-\mu_1\mu_2}{n}\right) + \left(\frac{1}{\mu_1^4}\right)\left(\frac{\mu_4-\mu_2^2}{n}\right). \tag{C.6}$$

So, substituting the results given in (C.2), (C.5) and (C.6) in (C.1) is obtained:

$$E\left(\frac{t}{\bar{X}} + \frac{\overline{X^2}}{2\left(\bar{X}\right)^2}\right)^2 \cong \frac{t^2}{\mu_1^2} + \frac{t\mu_2}{\mu_1^3} + \frac{\mu_2^2}{4\mu_1^4} + \frac{3t^2\sigma^2}{n\mu_1^4} + \frac{6t\mu_2\sigma^2}{n\mu_1^5} - \frac{3t}{\mu_1^4}\left(\frac{\mu_3-\mu_1\mu_2}{n}\right)$$
$$+\left(\frac{5\mu_2^2}{2\mu_1^6}\right)\frac{\sigma^2}{n} + \left(\frac{-2\mu_2}{\mu_1^5}\right)\left(\frac{\mu_3-\mu_1\mu_2}{n}\right) + \left(\frac{1}{4\mu_1^4}\right)\left(\frac{\mu_4-\mu_2^2}{n}\right).$$