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Solving Fractional Fredholm Integro-Differential Equations by Laguerre Polynomials

(Penyelesaian Pecahan Persamaan Pembezaan-Kamiran Fredholm menggunakan Polinomial Laguerre)

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ABSTRACT

The main purpose of this study was to present an approximation method based on the Laguerre polynomials to obtain the solutions of the fractional linear Fredholm integro-differential equations. This method transforms the integro-differential equation to a system of linear algebraic equations by using the collocation points. In addition, the matrix relation for Caputo fractional derivative of Laguerre polynomials is also obtained. Besides, some examples are presented to illustrate the accuracy of the method and the results are discussed.

Keywords: Fractional integro-differential equations; Fredholm integro-differential equations; Laguerre polynomials

ABSTRAK

Tujuan utama kajian ini adalah untuk mengemukakan kaedah penghampiran berdasarkan polinomial Laguerre untuk mendapatkan penyelesaian pecahan linear persamaan pembezaan-kamiran Fredholm. Kaedah ini menjelmakan persamaan pembezaan-kamiran ke sistem persamaan aljabar linear dengan menggunakan titik-titik kolokasi. Di samping itu, hubungan matriks untuk terbitan pecahan Caputo polinomial Laguerre juga diperoleh. Selain itu, beberapa contoh dibentangkan untuk menggambarkan ketepatan kaedah dan hasilnya dibincangkan.

Kata kunci: Persamaan pembezaan-kamiran Fredholm; persamaan pembezaan-kamiran pecahan; polinomial Laguerre

INTRODUCTION

Integro-differential equations play an important role in the modeling of numerous of physical phenomena from science and engineering. Hence, searching the exact and approximate solutions of integro-differential equations have attracted appreciable attention for scientists and applied mathematicians (Dzhumabaev 2018; Fairbairn & Kelmanson 2018; Hendi & Al-Qarni 2017; Kürkçü et al. 2017; Rahimkhani et al. 2017; Rohaninasab et al. 2018; Yüzbaşı & Karaçayır 2017). The fractional calculus represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering (Abbas et al. 2015). Since the fractional calculus has attracted much more interest among mathematicians and other scientists, the solutions of the fractional integro-differential equations have been studied frequently in recent years (Alkan & Hatipoglu 2017; Hamoud & Ghadle 2018a, 2018b; Ibrahim et al. 2015; Kumar et al. 2017; Ma & Huang 2014; Nemati et al. 2016; Ordokhani & Dehestani 2016; Parand & Nikarya 2014; Pedas et al. 2016; Shahooth et al. 2016; Turmetov & Abdullaev 2017; Wang & Zhu 2016; Yi et al. 2016). The methods that are used to find the solutions of the linear fractional Fredholm integro-differential equations are given as fractional pseudospectral integration matrices (Tang & Xu 2016), least squares with shifted Chebyshev polynomials (Mahdy et al. 2016; Mohammed 2014), least squares method using Bernstein polynomials (Oyedepo et al. 2016), fractional residual power series method (Syam 2017), Taylor matrix method (Gülsu et al. 2013), reproducing kernel Hilbert space method (Bushnaq et al. 2016), second kind Chebyshev wavelet method (Setia et al. 2014), open Newton method (Al-Jamal & Rawashdeh 2009), modified Homotopy perturbation method (Elbeleze et al. 2016), Sinc collocation method (Emiroglu 2015).

Laguerre polynomials are used to solve some integer order integro-differential equations. These equations are given as Altarelli-Parisi equation (Kobayashi et al. 1995), Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equation (Schoeffel 1999), Pantograph-type Volterra integro-differential equation (Yüzbaşı 2014), linear Fredholm integro-differential equation (Baykus Savasaneril & Sezer 2016; Gürbüz et al. 2014), linear integro-differential equation (Al-Zubaidy 2013), parabolic-type Volterra partial integro-differential equation (Gürbüz & Sezer 2017a), nonlinear partial integro-differential equation (Gürbüz & Sezer 2017b), delay partial functional differential equation (Gürbüz & Sezer 2017c). Besides, Laguerre polynomials are used to solve the fractional integro-differential equation (Mahdy & Shwayyea 2016).

Consider the following linear fractional Fredholm integro-differential equation,

$$D^{\alpha}y(x)=g(x)+\int_0^1 K(x,t)y(t)\,dt,\; n-1<\alpha<n, n\in N, \tag{1}$$

with the following initial conditions:

$$y^{(j)}(0)=c_j \quad j=0,1,\ldots,n-1. \tag{2}$$

where $D^{\alpha}y(x)$ indicates the Caputo fractional derivative of $y(x)$; $K(x, t)$ and $g(x)$ are given functions, x and t are real variables varying in the interval [0, 1] and $y(x)$ is the unknown function to be determined. Now, we give the definition and the basic properties of the Caputo fractional derivative as follows:

Definition (Podlubny 1999). The Caputo fractional differentiation operator D^{α} of order α is defined as follows:

$$D^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt,\; \alpha>0$$

where $n-1<\alpha<n$, $n\in\mathbb{Z}^{+}$. Besides, Caputo fractional derivative of a constant function is zero and Caputo fractional differentiation operator is linear (Herrman 2014).

The aim of this study was to give an approximate solution of the problem (1)-(2) in the form,

$$y(x)\cong y_N(x)=\sum\nolimits_{i=0}^{N} a_i L_i(x) \tag{3}$$

where a_i are unknown coefficients; N is chosen any positive integer such that $N\geq n$; and $L_i(x)$, $(i=0,1,\ldots,N)$ are the Laguerre polynomials of order i defined by Bell (1968),

$$L_i(x)=\sum_{k=0}^{i}(-1)^k\frac{i!}{(i-k)!(k!)^2}x^k.$$

MAIN MATRIX RELATIONS

In this section, we will construct the matrix forms of each term of (1). Firstly, we can write the approximate solution (3) in the matrix form,

$$y_N(x)=\mathbf{L}(x)\mathbf{A}, \tag{4}$$

where

$$\mathbf{L}(x)=[L_0(x)\;\,L_1(x)\;\,\ldots\;\,L_N(x)] \text{ and } \mathbf{A}=[a_0\;\,a_1\;\,\ldots\;\,a_N]^T.$$

Now, we will state a theorem that gives the Caputo fractional derivative of Laguerre polynomials in terms of Laguerre polynomials:

Theorem: Let $L_i(x)$ be Laguerre polynomial of order i, then the Caputo fractional derivative of $L_i(x)$ in terms of Laguerre polynomials are found as follows:

$$D^{\alpha}L_i(x)=x^{-\alpha}\sum_{k=\lceil\alpha\rceil}^{i}\sum_{j=0}^{k}(-1)^{j+k}\frac{k!}{\Gamma(k+1-\alpha)}\binom{i}{k}\binom{k}{j}L_j(x),$$
$$i=\lceil\alpha\rceil,\lceil\alpha\rceil+1,\ldots \tag{5}$$

for $0<x<\infty$, where $[\alpha]$ denotes for the ceiling function which is the smallest integer greater than or equal to α.

Proof Let us begin derivating the Laguerre polynomials with the definition of them,

$$D^{\alpha}L_i(x)=D^{\alpha}\left\{\sum_{k=0}^{i}(-1)^k\frac{i!}{(i-k)!(k!)^2}x^k\right\}$$

By the linearity of Caputo fractional derivative, we get

$$D^{\alpha}L_i(x)=\sum_{k=0}^{i}(-1)^k\frac{i!}{(i-k)!(k!)^2}D^{\alpha}(x^k).$$

Using the Caputo fractional derivative of x^k

$$D^{\alpha}x^k=\begin{cases}0 & k<\lceil\alpha\rceil \\ \dfrac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}x^{k-\alpha}, & k\geq\lceil\alpha\rceil\end{cases},\quad k=0,1,2,\ldots$$

we obtain,

$$D^{\alpha}L_i(x)=\sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^k}{\Gamma(k+1-\alpha)}\binom{i}{k}x^{k-\alpha},\quad i=\lceil\alpha\rceil,\lceil\alpha\rceil+1,\ldots$$

At this step, by taking $x^{-\alpha}$ out of the series and using the Laguerre series of the function x^k given in the reference by Lebedev (1972),

$$x^k=k!\sum_{j=0}^{k}(-1)^j\binom{k}{j}L_j(x),\; 0<x<\infty,\; k=0,1,2\;\ldots,$$

we have the relation (5).

MATRIX REPRESENTATION FOR THE DIFFERENTIAL PART

Now, we will write the matrix form of the differential part of the (1). It is obviously seen that

$$D^{\alpha}\mathbf{L}(x)=[D^{\alpha}L_0(x)\;\,D^{\alpha}L_1(x)\;\ldots\;D^{\alpha}L_N(x)] \tag{6}$$

The right hand side of this equation can be expressed as,

$$D^{\alpha}\mathbf{L}(x)=x^{-\alpha}\mathbf{L}(x)\mathbf{S}_{\alpha} \tag{7}$$

where $\mathbf{S}_{\alpha}$ is an $(N+1)$ dimensional square matrix denoted by,

$$\mathbf{S}_{\alpha}=\begin{bmatrix}0 & A_{1,1} & \binom{1}{0}A_{1,2}+\binom{2}{0}A_{2,2} & & \sum_{j=1}^{N}\binom{j}{0}A_{j,N} \\ 0 & -A_{1,1} & -\left[\binom{1}{1}A_{1,2}+\binom{2}{1}A_{2,2}\right] & \cdots & -\sum_{j=1}^{N}\binom{j}{1}A_{j,N} \\ 0 & 0 & \binom{2}{2}A_{2,2} & & \sum_{j=1}^{N}\binom{j}{2}A_{j,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^N\binom{N}{N}A_{N,N}\end{bmatrix}.$$

Here, the $A_{k,i}$ terms in the entries of $\mathbf{S}_\alpha$ are defined as:

$$\mathbf{A}_{k,i} = \begin{cases} (-1)^k \frac{k!}{\Gamma(k+1-\alpha)}\binom{i}{k}, & \text{if } i \geq k \geq \lceil \alpha \rceil \\ 0, & \text{otherwise} \end{cases}.$$

Then, using (4) and (7) the differential part of the (1) can be expressed as,

$$D^\alpha y(x) = D^\alpha \mathbf{L}(x)\mathbf{A} = x^{-\alpha}\mathbf{L}(x)\mathbf{S}_\alpha \mathbf{A}. \tag{8}$$

MATRIX REPRESENTATION FOR THE INTEGRAL PART

Let us define the integral part of the equation by,

$$I(x) = \int_0^1 K(x,t)y(t)dt$$

and then we assume the kernel function can be expanded on the truncated series of Laguerre polynomials with respect to the second variable t as,

$$K(x,t) \cong \sum_{n=0}^{N} k_n(x)L_n(t).$$

Therefore, the kernel can be written in the matrix form,

$$K(x,t) = \mathbf{K}(x)\mathbf{L}^{\mathrm{T}}(t)$$

where $\mathbf{K}(x) = [k_0(x) \;\; k_1(x) \;\; \ldots \;\; k_N(x)]$. Hence, the matrix representation of the integral becomes

$$I(x) = \int_0^1 \mathbf{K}(x)\mathbf{L}^{\mathrm{T}}(t)\mathbf{L}(t)\mathbf{A}dt = \mathbf{K}(x)\mathbf{Q}\,\mathbf{A} \tag{9}$$

where $\mathbf{Q} = [q_{ij}]$ and $q_{ij} = \int_0^1 L_i(t)L_j(t)dt, \quad i, j = 0,1,\ldots,N.$

MATRIX RELATION FOR THE CONDITIONS

The relation between $\mathbf{L}(x)$ and its derivatives of integer order is given by Yüzbaşı (2014) as,

$$D^i\mathbf{L}(x) = \mathbf{L}(x)\mathbf{M}^i, \; i = 0,1,2\ldots \tag{10}$$

where the matrix $\mathbf{M}$ is given as

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & -1 & & -1 \\ 0 & 0 & -1 & \cdots & -1 \\ 0 & 0 & 0 & & -1 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & & 0 \end{bmatrix}.$$

Using the relation (10) the corresponding matrix forms of the conditions defined in (2) can be written as,

$$y^j(0) = \mathbf{L}(0)\mathbf{M}^j\mathbf{A} = c_j, \; j = 0,1,\ldots,n-1 \tag{11}$$

Here, the matrix $\mathbf{L}(0)\mathbf{M}^j$ is named as $\mathbf{U}_{\mathrm{j}}$ where it is an $1 \times (N+1)$ dimensional matrix. Hence, the equation (11) becomes,

$$\mathbf{U}_{\mathrm{j}}\mathbf{A} = c_j, \quad j = 0.1,\ldots,n-1.$$

METHOD OF SOLUTION

To obtain the approximate solution of (1), we compute the unknown coefficients by using the following collocation method. Firstly, let us substitute the matrix forms (8) and (9) into (1) and thus we obtain the matrix equation,

$$D^\alpha y(x) = g(x) + \int_0^1 K(x,t)y(t)dt$$

$$x^{-\alpha}\mathbf{L}(x)\mathbf{S}_\alpha\mathbf{A} = g(x) + \mathbf{K}(\mathrm{x})\mathbf{Q}\,\mathbf{A} \tag{12}$$

By substituting the collocation points $x_s > 0$ into (12), we have a system of matrix equations

$$x_s^{-\alpha}\mathbf{L}(x_s)\mathbf{S}_\alpha\mathbf{A} = g(x_s)\mathbf{Q}\,\mathbf{A}, \quad s = 0,1,\ldots,N. \tag{13}$$

This system can be written in the compact forms:

$$\mathbf{X}_\alpha\mathbf{L}\mathbf{S}_\alpha\mathbf{A} = \mathbf{G} + \mathbf{K}\mathbf{Q}\mathbf{A}$$

or

$$\{\mathbf{X}_\alpha\mathbf{L}\mathbf{S}_\alpha - \mathbf{K}\mathbf{Q}\}\mathbf{A} = \mathbf{G} \tag{14}$$

where

$$\mathbf{X}_\alpha = \begin{bmatrix} x_0^{-\alpha} & 0 & & 0 \\ 0 & x_1^{-\alpha} & \cdots & \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_N^{-\alpha} \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}(x_0) \\ \mathbf{L}(x_1) \\ \vdots \\ \mathbf{L}(x_N) \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}(x_0) \\ \mathbf{K}(x_1) \\ \vdots \\ \mathbf{K}(x_N) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

Denoting the expression in parenthesis of (14) by $\mathbf{W}$, the fundamental matrix equation for (1) is reduced to $\mathbf{WA} = \mathbf{G}$ which corresponds to a system of $(N+1)$ linear algebraic equations with unknown Laguerre coefficients $a_0, a_1, \ldots, a_N$.

Finally, to obtain the solution of (1) under the conditions (2), we replace the n rows of the augmented matrix $[\mathbf{W}; \mathbf{G}]$ with the rows of the augmented matrix $[\mathbf{U}_{\mathrm{j}}; c_j]$. In this way, the Laguerre coefficients are determined by solving the new linear algebraic system.

NUMERICAL EXAMPLES

In this section, we apply the method to two examples and we have performed all of the numerical computations using Mathcad 15. We also use the collocation points by using the formula $x_s = \left[1 - \cos\left(\frac{(s+1)\pi}{N+1}\right)\right]/2.$

Example 1 Consider the following fractional integro-differential equation,

$$D^{\frac{1}{2}}y(x)=\frac{\frac{8}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}}{\sqrt{\pi}}+\frac{x}{12}+\int_0^1 xty(t)dt,\qquad 0\leq x,\ t\ \leq 1,$$

subject to $y(0) = 0$ with the exact solution $y(x) = x^2 - x$.

The collocation points for $N = 2$ becomes $x_0 = 0.25$, $x_1 = 0.75, x_2 = 1$. The main matrix equation of this problem is given by

$$\{\mathbf{X}_{1/2}\mathbf{L}\mathbf{S}_{1/2}-\mathbf{K}\mathbf{Q}\}\mathbf{A}=\mathbf{G}$$

where the matrices are,

$$\mathbf{X}_{1/2}=\begin{bmatrix}2&0&0\\0&\frac{2\sqrt{3}}{3}&0\\0&0&1\end{bmatrix},\quad \mathbf{L}=\begin{bmatrix}1&\frac{3}{4}&\frac{17}{32}\\1&\frac{1}{4}&-\frac{7}{32}\\1&0&-\frac{1}{2}\end{bmatrix},$$

$$\mathbf{G}=\begin{bmatrix}\frac{1}{48}&-&\frac{2}{3\sqrt{\pi}}\\&\frac{1}{16}&\\\frac{1}{12}&+&\frac{2}{3\sqrt{\pi}}\end{bmatrix},\quad \mathbf{K}=\begin{bmatrix}\frac{1}{4}&-\frac{1}{4}&0\\\frac{3}{4}&-\frac{3}{4}&0\\1&-1&0\end{bmatrix},$$

$$\mathbf{Q}=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{6}\\1&\frac{1}{3}&\frac{5}{24}\\\frac{1}{6}&\frac{5}{24}&\frac{13}{60}\end{bmatrix}.$$

Also, the matrix forms of the initial condition is calculated as $\mathbf{U}_0 = \mathbf{L}(0) = [1\ \ 1\ \ 1]$.

By solving this system we get $a_0 = 1, a_1 = -3, a_2 = 2$. When we substitute the determined coefficients into (3), we get the exact solution.

This problem is also solved by Mahdy et al. (2016), Mohammed (2014) and Oyedepo et al. (2016). Mahdy et al. (2016) and Mohammed (2014) had found an approximate solution for $N = 7$ and $N = 5$, respectively; but they didn't state the numerical results of the errors of their methods. They had said that their solutions were in agreement with the exact solution according to the graphs of the solution. Hovewer, it is not enough to say that they have found the exact solution, since their graphs are drawn at 0.1 scale. Oyedepo et al. (2016) had found the approximate solution with the maximum absolute error 4.10^{-5} and 1.10^{-4} by the standard least squares method and the perturbed least squares method, respectively. By the proposed method, we have found the exact solution of the problem for $N = 2$. Apparently, our method is better than the other methods.

Example 2 Consider the following fractional integro-differential equation

$$D^{\frac{5}{3}}y(x)=\frac{3\sqrt{3}\Gamma(2/3)x^{\frac{1}{3}}}{\pi}-\frac{x^2}{5}-\frac{x}{4}+\int_0^1(xt+x^2t^2)y(t)dt,$$

$$0\leq x,\ t\leq 1,$$

subject to $y(0) = 0$ with the exact solution $y(x) = x^2$.

The collocation points for $N = 2$ becomes $x_0 = 0.25$, $x_1 = 0.75, x_2 = 1$. The main matrix equation of this problem is given by,

$$\{\mathbf{X}_{5/3}\mathbf{L}\mathbf{S}_{5/3}-\mathbf{K}\mathbf{Q}\}\mathbf{A}=\mathbf{G}$$

where **L** and **Q** matrices are the same as in the Example 1 and the others are given by,

$$\mathbf{X}_{5/3}=\begin{bmatrix}4^{5/3}&0&0\\0&\left(\frac{4}{3}\right)^{5/3}&0\\0&0&1\end{bmatrix},\quad \mathbf{K}=\begin{bmatrix}\frac{3}{8}&-\frac{1}{2}&\frac{1}{8}\\\frac{15}{8}&-3&\frac{9}{8}\\3&-5&2\end{bmatrix},$$

$$\mathbf{G}=\begin{bmatrix}\frac{3\sqrt{3}\left(4^{2/3}\right)\Gamma\left(\frac{2}{3}\right)}{4\pi}-\frac{3}{40}\\\frac{\left(3^{11/6}\right)\left(4^{2/3}\right)\Gamma\left(\frac{2}{3}\right)}{4\pi}-\frac{3}{10}\\\frac{3\sqrt{3}\Gamma\left(\frac{2}{3}\right)}{\pi}-\frac{9}{20}\end{bmatrix}.$$

Also, the matrix forms of the initial condition is calculated as $\mathbf{U}_0 = \mathbf{L}(0)$.

By solving this system, we get $a_0 = 2, a_1 = -4, a_2 = 2$. When we substitute the determined coefficients into (3), we get the exact solution.

This problem is also solved by Mahdy et al. (2016) and Mohammed (2014), and they had found an approximate solution for $N = 7$ and $N = 5$, respectively. As in previous example, we do not have any numerical results to compare the methods. By the proposed method, we have found the exact solution of the problem for $N = 2$. Apparently, our method is better than the other methods.

Example 3 Consider the following fractional integro-differential equation

$$D^{\frac{1}{2}}y(x)=\frac{2\sqrt{x}}{\sqrt{\pi}}+\frac{3x\sqrt{\pi}}{4}-\frac{9}{10}+\int_0^1 y(t)\,dt,\qquad 0\leq x,\ t\leq 1,$$

subject to $y(0) = 0$ with the exact solution $y(x) = x^{3/2} + x$.

We write the main matrix equation of this problem as follows:

$$\{\mathbf{X}_{1/2}\mathbf{L}\mathbf{S}_{1/2}-\mathbf{K}\mathbf{Q}\}\mathbf{A}=\mathbf{G}.$$

The absolute errors of our method are compared with the second kind Chebyshev wavelet method (Setia et al. 2014) in Table 1. It is seen that our method gives better results than the other method.

Example 4 Consider the following fractional integro-differential equation

$$D^{\frac{5}{6}}y(x)=\frac{3x^{\frac{1}{6}}\Gamma(5/6)(-91+216x^2)}{91\pi}+(5-2e)x+\int_0^1 xe^t y(t)dt,$$

$$0\leq\ x,\ t\leq 1,$$

subject to $y(0) = 0$ with the exact solution $y(x) = x - x^3$.

TABLE 1. Comparison of the absolute errors for Example 3

x	Chebyshev wavelet method for 16 terms	Chebyshev wavelet method for 32 terms	Our method for 6 terms ***N* = 5**	Our method for 8 terms ***N* = 7**
0.1	4.4×10^{-3}	1.7×10^{-3}	4.1×10^{-4}	9.5×10^{-5}
0.2	6.1×10^{-3}	1.7×10^{-3}	1.8×10^{-4}	1.6×10^{-4}
0.3	6.2×10^{-3}	1.8×10^{-3}	3.2×10^{-4}	2.4×10^{-4}
0.4	5.7×10^{-3}	1.9×10^{-3}	8.0×10^{-4}	5.8×10^{-5}
0.5	4.8×10^{-3}	2.0×10^{-3}	7.4×10^{-4}	6.7×10^{-5}
0.6	6.5×10^{-3}	2.2×10^{-3}	3.4×10^{-4}	2.3×10^{-4}
0.7	7.4×10^{-3}	2.3×10^{-3}	1.6×10^{-4}	2.2×10^{-4}
0.8	7.8×10^{-3}	2.5×10^{-3}	5.2×10^{-4}	7.8×10^{-5}
0.9	7.6×10^{-3}	2.6×10^{-3}	9.7×10^{-4}	2.2×10^{-4}
1.0	7.0×10^{-3}	2.7×10^{-3}	3.5×10^{-4}	1.0×10^{-4}

TABLE 2. Comparison of the absolute errors for Example 4

x	Standard least squares method	Perturbed least squares method	Our method ***N* = 6**
0.1	6.3036×10^{-5}	0.7034×10^{-5}	6.7988×10^{-8}
0.2	2.5659×10^{-5}	3.3421×10^{-5}	2.4063×10^{-7}
0.3	6.8668×10^{-5}	1.6037×10^{-6}	5.0604×10^{-7}
0.4	3.2130×10^{-5}	2.2706×10^{-5}	8.5722×10^{-7}
0.5	4.7716×10^{-5}	3.7106×10^{-5}	1.2901×10^{-6}
0.6	5.1213×10^{-5}	3.9193×10^{-5}	1.8018×10^{-6}
0.7	4.0208×10^{-5}	2.6563×10^{-5}	2.3903×10^{-6}
0.8	1.2286×10^{-5}	3.1847×10^{-6}	3.0533×10^{-6}
0.9	3.4964×10^{-5}	5.2455×10^{-5}	3.7890×10^{-6}
1.0	1.0395×10^{-5}	1.2365×10^{-4}	4.5965×10^{-6}

We write the main matrix equation of this problem as follows:

$$\{\mathbf{X}_{5/6}\mathbf{L}\mathbf{S}_{5/6} - \mathbf{K}\mathbf{Q}\}\mathbf{A} = \mathbf{G}.$$

This problem is also solved by Mohammed (2014) and Oyedepo et al. (2016). To compare, Mohammed hadn't found the exact solution. He had found an approximate solution for $N = 5$ but he didn't state the numerical results of the errors of the method. The absolute errors of our method are compared with the standard least squares method (Oyedepo et al. 2016) and perturbed least squares method (Oyedepo et al. 2016) in Table 2. It is seen that our method gives better results than the other methods.

CONCLUSION

In this study, a collocation method based on Laguerre polynomials has been introduced for solving the fractional linear Fredholm integro-differential equations. For this purpose, the matrix relation for the Caputo fractional derivative of the Laguerre polynomials has been obtained for the first time in the literature. Using these relations and suitable collocation points, the integro-differential equation has been transformed into a system of algebraic equations. The method is faster and simpler than the other methods in the literature.

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