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Defaultable Bond Pricing under the Jump Diffusion Model with Copula Dependence Structure

(Penentuan Harga Bon Boleh Mungkir di Bawah Model Resapan Lompat dengan Struktur Kebersandaran Kopula)

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ABSTRACT

We study the pricing of a defaultable bond under various dependence structure captured by copulas. For that purpose, we use a bivariate jump-diffusion process to represent a bond issuer's default intensity and the market short rate of interest. We assume that each jump of both variables occur simultaneously, and that their sizes are dependent. For these simultaneous jumps and their sizes, a homogeneous Poisson process and three copulas, which are a Farlie-Gumbel-Morgenstern copula, a Gaussian copula, and a Student t-copula are used, respectively. We use the joint Laplace transform of the integrated risk processes to obtain the expression of the defaultable bond price with copula-dependent jump sizes. Assuming exponential marginal distributions, we compute the zero coupon defaultable bond prices and their yields using the three copulas to illustrate the bond. We found that the bond price values are the lowest under the Student-t copula, suggesting that a dependence structure under the Student-t copula could be a suitable candidate to depict a riskier environment. Additionally, the hypothetical term structure of interest rates under the risky environment are also upward sloping, albeit with yields greater than 100%, reflecting a higher compensation required by investors to lend funds for a longer period when the financial market is volatile.

Keywords: Bivariate jump-diffusion model; credit risk; default intensity; short rate; zero coupon bond

ABSTRAK

Kertas ini mengkaji penentuan harga bon boleh mungkir dengan kadar faedah pendek dan nilai keamatan ingkar penerbit bon, dengan struktur kebersandaran yang diwakili oleh kopula. Untuk tujuan itu, proses resapan-lompat bivariat digunakan untuk mewakili proses keamatan ingkar penerbit bon dan kadar faedah pendek pasaran. Setiap lompatan oleh kedua-dua pemboleh ubah diandaikan berlaku serentak, dan saiznya adalah bersandaran antara satu sama lain. Bagi mewakili proses lompatan serentak dan struktur kebersandaran saiznya, proses Poisson yang homogen dan tiga kopula, iaitu kopula Farlie-Gumbel-Morgenstern, Gaussian, dan student-t digunakan. Transformasi Laplace tercantum bagi proses risiko bersepadu digunakan untuk mendapatkan persamaan harga bon boleh mungkir dengan saiz lompatan faktor yang bersandar dengan struktur kopula. Harga bon boleh mungkir tanpa kupon dan kadar hasilnya dihitung di bawah tiga jenis kopula dengan taburan marginal eksponen untuk mewakili kebersandaran antara kedua-dua faktor. Kajian mendapati bahawa nilai harga bon adalah yang paling rendah apabila faktor kebersandaran digambarkan oleh kopula student-t, yang menunjukkan bahawa struktur kebersandaran di bawah kopula student-t adalah lebih sesuai untuk menggambarkan persekitaran yang berisiko berbanding kopula FGM dan Gaussian. Di samping itu, walaupun struktur masa kadar faedah bagi jangka panjang di bawah persekitaran yang berisiko juga menunjukkan pola menaik, kadar hasil yang melebihi 100%, mencerminkan situasi bahawa pelabur memerlukan pampasan yang lebih tinggi bagi aktiviti meminjamkan dana untuk tempoh yang lebih lama apabila situasi pasaran kewangan adalah tidak menentu.

Kata kunci: Bon sifar kupon; kadar keamatan mungkir; kadar pendek; model resapan-lompat bivariat; risiko kredit

INTRODUCTION

After a decade since the 2008 Global Financial Crisis (GFC), the overall measures of US household financial wellness show that many households remain vulnerable with increasing auto loan, student debts as well as credit card debt although the household debt to GDP ratio has reduced by 19%. An extended period of low interest rates has also witnessed global nonfinancial corporate debt doubled, hitting $66 trillion for the past 10 years. Furthermore, the global public debt has soared to $60 trillion, at an average of 105% of GDP among the advanced economies and 46% among the emerging economies (Lund et al. 2018). Therefore, it is necessary to develop prudent quantitative models for corporate bonds pricing given the extent of the interest rate risk exposure to the economic agents, despite the allegations that mathematical models were the reason for the losses borne during the GFC (Donelly & Embrechts 2010; Salmon 2009; Stewart 2012).

The two classes of models in credit risk evaluation for corporate debt pricing are the structural model and the reduced form model. Under the structural approach, a firm's liabilities are viewed as contingent claims issued against its assets, with all the payoffs to the firm's liabilities in bankruptcy completely specified (Black & Cox 1976; Merton 1974). In other words, bankruptcy is viewed as the event when the firm's asset value hits a pre-specified boundary. The view undertaken in this class of models was then ameliorated in Hull and White (1995) as well as Longstaff and Schwartz (1995), whereby the cash flows to risky debt were exogenously specified as a given fraction of each promised dollar in the event of bankruptcy. This perspective would be useful when considering the complex priority structure of payoffs made to a firm's liabilities. Ruf and Scherer (2011) then computed the bond prices following a structural default model with jumps using Monte Carlo simulation based on a Brownian bridge algorithm.

Rather than examining the firm's internal structure whose complete information is not available to the external parties as in the structural approach, we work under the reduced form approach which requires a different set of information that is less refined. We refer the readers to Jarrow and Protter (2004) for a thorough discussion on the comparison between the structural and the reduced form models. Under the reduced form approach, we observe the information generated by a vector of state variables and the firm's default time. The firm's default time was generated by a Cox process with an intensity process that also depends on the state variables. Lando (1998) prepared a convenient framework that allows for dependencies between default intensities and state variables, whereby the Cox process was used to model the (stopping) time when the rating changed until the issuer went default in the last state of a generalized K-states Markovian model. In one of the earliest papers to promote the term 'reduced-form' approach, Duffie and Singleton (1999) treated default as an unpredictable event governed by the external hazard rate process. A contingent claim that is subject to default risk can be priced just like the default-free claim simply by replacing the short rate with the default-adjusted short rate process under an equivalent martingale measure in an arbitrage free framework. This approach was then extended in Kijima (2000) which considered the possibility of default-event triggers that cause joint default, and in Jarrow and Yu (2001) which introduced the concept of counterparty risk to capture the economy-wide and inter-firm linkages.

Previous studies of the reduced form approach have taken several directions in researchers' attempts to incorporate default correlation and multiple defaults (Brigo & Chourdakis 2009; Herbertsson et al. 2011; Jang 2007; Ma & Kim 2010). The reduced form model promoted in Jarrow and Turnbull (1995) was further developed in Jarrow et al. (1997), whereby the bankruptcy process was modelled as a continuous time Markov process with discrete state space representing the firm's credit ratings. By combining the structural and the reduced form approaches, the authors specified the credit event exogenously and allowed the bankruptcy assumptions to be imposed only on observables (i.e. the firm's credit ratings) as opposed to the firm's asset values. Another hybrid example can also be found in Hyong-Chol and Ning (2005), which assumed an expected and unexpected default following stochastic default intensity and provided an explicit pricing formula for defaultable bond and credit default swap (CDS).

Another approach to incorporate default dependence between related parties is through the use of copulas (Giesecke 2004; Jouanin et al. 2001; Li 2000; Schonbucher & Schubert 2001). The use of Farlie-Gumbel-Mogenstern (FGM) copula to price a CDS using the multivariate shot noise process was studied in Ma and Kim (2010) and extended by Mohd Ramli and Jang (2015) by adding a diffusion term on the shot noise process making it a multivariate jump-diffusion process. A study by Jang and Mohd Ramli (2018) applied the joint survival probability expression to examine the effect of the jump-diffusion process on a social benefit scheme consisting of life insurance, as well as unemployment/disability and retirement benefits. Hence in this paper, we propose the jump diffusion process to represent the risky environment of a financial market via the short interest rate and the bond issuer's default intensity. Due to the strong evidence that the default intensity varying with the business cycle (Kijima 2000), we aim to examine the bond price behaviour under the risky environment represented by the jump diffusion process using three types of two-tailed copula, which also allows flexibility in modelling the variables and its combined effect.

The remaining of the paper is organized as follows: Next section defines the bivariate jump-diffusion process for short rate and firm's default intensity, and the three copulas used to capture the dependence between the jump sizes of the variables, i.e. the FGM, Gaussian and student's t-copula. The expression of the bond price obtained from the results in Mohd Ramli and Jang (2015) was also presented. This is then followed by a numerical example in subeseqent section showing the computation and the comparison of bond prices and their yields under the three copulas, showcasing the advantage of the Student-t copula over the other two copulas considered. Final section concludes the paper.

METHODS

This section describes the quantitative model used to represent the important variables used in the defaultable bond pricing, and the three copulas used to illustrate the dependence structure.

BIVARIATE JUMP DIFFUSION MODEL

For $i = 1$ (bond issuer's default intensity) and $i = 2$ (short rate), the jump-diffusion Cox-Ingersoll-Ross (CIR) process considered has the following structure:

$$d\lambda^{(i)}(t) = c^{(i)}\big(b^{(i)} + a^{(i)}\lambda^{(i)}(t)\big)dt + \sigma^{(i)}\sqrt{\lambda^{(i)}(t)}dW^{(i)}(t)$$

$$+ dL^{(i)}(t)$$

$$\text{with} \quad L^{(i)}(t) = \sum_{h=1}^{M^{(i)}(t)} Y_h^{(i)}, \qquad (1)$$

where $c^{(i)}b^{(i)}$ represents the long-term mean level of the short rate or default intensity; $c^{(i)}a^{(i)}$ represents the drift coefficient, which is the speed at which the factor is driven back to its long-term mean, with $c^{(i)}a^{(i)}<0$; $\sigma^{(i)}$ is the diffusion coefficient; and $W^{(i)}(t)$ is a standard Brownian motion governing the process.

We define $L^{(i)}(t)$ as a pure jump process in which $M^{(i)}(t)$ is the number of jumps, representing the total number of events up to time t and $Y_h^{(i)}$, $h = 1, 2, \cdots, M^{(i)}(t)$ is their sizes. The point process $M^{(i)}(t)$ with average $\rho^{(i)}$ is independent of the vector sequence of jump sizes. The jump occurrences are assumed to be simultaneous for both processes and that their sizes are independent and identically distributed (i.i.d.) with distribution function $F^{(i)}(y)$. The process needs to satisfy the usual f eller condition given by $\frac{2c^{(i)}b^{(i)}}{a^{(i)}} > \sigma^{(i)2}$ \$ to ensure $\lambda^{(i)}(t) > 0$. Note that, by setting $\rho^{(i)} = 0$, the jump diffusion process becomes the celebrated Cox-ingersoll-Ross (CiR) process (Cox et al. 1985), while setting $\sigma^{(i)} = 0$ gives us the shot noise process (Jang 2007; Ma & Kim 2010). The presence of the diffusion and jump terms make the jump diffusion process a more suitable candidate to represent a risky economic environment, relative to the CiR model (1985) and the shot noise processes.

Defaultable Bond Price Expression

Using the multivariate joint Laplace transform proposed in Mohd Ramli and Jang (2015), we obtain the expression for a defaultable bond price, $\mathbb{E}\left[e^{-\int_0^t (d_s + r_s)ds}\right]$,whereby the d_s denotes the default rate of the issuer and r_s denotes the interest rates. The joint Laplace transform is useful to compute the survival probability of an insured life given an insurance contract, as well as the bond price.

As there are two variables involved, we let $n = 2$ whereby i =1 is the obligor's default intensity and i = 2 is the market short rate. f or each process, we also define its cumulative hazard process, $\Lambda^{(i)}(t) = \lambda^{(i)}(s)\, ds$, which is the sum of risks related to the process i that we encounter from time 0 to t. a s proposed and proven in Corollary 2 of Mohd Ramli and Jang (2015), the joint Laplace transform up to time t described by the filtration $\mathcal{G}_t = \sigma\{(\lambda^{(1)}(s), \lambda^{(2)}(s)): s \leq t\} \in \Re^n$ is given by the following proposition:

Proposition 1 Considering constants $\alpha^{(i)} \geq 0$ and $\gamma^{(i)} \geq 0$ for i = 1, 2, the joint Laplace transform of the vector $(\Lambda^{(1)}(t), \Lambda^{(2)}(t), \lambda^{(1)}(t), \lambda^{(2)}(t))$ is given by

$$\mathbb{E}\left[\exp\left\{-\sum_{i=1}^{2}\gamma^{(i)}\Lambda^{(i)}(t) + \alpha^{(i)}\lambda^{(i)}(t)\right\}\Big|\mathcal{G}_0\right]$$

$$= \prod_{i=1}^{2}\left[\mathrm{H}^{(i)}(t)^{\frac{2c^{(i)}b^{(i)}}{\sigma^{(i)2}}}\right]\exp\Big(-\sum_{i=1}^{2}\mathrm{G}^{(i)}(t)\lambda^{(i)}(0) + \rho^{(i)}$$

$$\left[\int_0^t 1 - \hat{c}\big(\mathrm{G}^{(1)}(s),\mathrm{G}^{(2)}(s)\big)\right]ds\Big) \qquad (2)$$

where t > 0, with

$$\mathrm{g}^{(i)}(t) = \frac{\alpha^{(i)}[(\mathrm{D}^{(i)} + \mathrm{c}^{(i)}\mathrm{a}^{(i)}) + \mathrm{D}^{(i)} - \mathrm{c}^{(i)}\mathrm{a}^{(i)}\exp\{-\mathrm{D}^{(i)}t\}] +}{\sigma^{(i)(2)}\alpha^{(i)}(1 - \exp\{-\mathrm{D}^{(i)}t\}) + (\mathrm{D}^{(i)} - \mathrm{c}^{(i)}\mathrm{a}^{(i)}) +}$$

$$\frac{2\gamma^{(i)}(1 - \exp\{-\mathrm{D}^{(i)}t\})}{(\mathrm{D}^{(i)} + \mathrm{c}^{(i)}\mathrm{a}^{(i)})\exp\{-\mathrm{D}^{(i)}t\}}$$

$$\mathrm{H}^{(i)}(t) = \frac{2\mathrm{D}^{(i)}\exp\left\{-\frac{(\mathrm{D}^{(i)} + \mathrm{c}^{(i)}\mathrm{a}^{(i)})}{2}t\right\}}{\sigma^{(i)^2}\alpha^{(i)}(1 - \exp\{-\mathrm{D}^{(i)}t\}) + (\mathrm{D}^{(i)} - \mathrm{c}^{(i)}\mathrm{a}^{(i)}) + (\mathrm{D}^{(i)} + \mathrm{c}^{(i)}\mathrm{a}^{(i)})\exp\{-\mathrm{D}^{(i)}t\}}$$

$$\mathrm{D}^{(i)} = \sqrt{(\mathrm{c}^{(i)}\mathrm{a}^{(i)})^2 + 2\gamma^{(i)}\sigma^{(i)2}}$$

and

$$\hat{c}\big(\xi^{(1)},\xi^{(2)}\big) = \int_0^\infty\int_0^\infty \exp\left\{-\sum_{i=1}^{2}\xi^{(i)}y_i\right\}\frac{\partial^2 C\big(F_{Y^{(1)}}(y_1),F_{Y^{(2)}}(y_2)\big)}{\partial y_1\partial y_2}$$

$$dy_1\, dy_2\,.$$

Note that the bivariate joint Laplace transform contains a random element given by the term $\hat{c}\big(\xi^{(1)},\xi^{(2)}\big)$, which incorporates the jump-size distribution and the dependence structure between the jump sizes of the variables. Setting $\alpha^{(i)} = 0$ in (2), we obtain the bond price expression, as presented in the following proposition:

Proposition 2 The joint Laplace transform of the vector $(\Lambda^{(1)}(t), \Lambda^{(2)}(t))$ is given by

$$\mathbb{E}\left[\exp\left\{-\sum_{i=1}^{2}\gamma^{(i)}\Lambda^{(i)}(t)\right\}\Big|\mathcal{G}_0\right]$$

$$= \prod_{i=1}^{2}\left[\mathrm{J}^{(i)}(t)^{\frac{2c^{(i)}b^{(i)}}{\sigma^{(i)2}}}\right]\exp\Big(-\sum_{i=1}^{2}\mathrm{K}^{(i)}(t)\lambda^{(i)}(0) + \rho^{(i)}$$

$$\rho^{(i)}\left[\int_0^t 1 - \hat{c}(K(s),K(s))\right]ds\Big), \qquad (3)$$

where t > 0, with

$$\mathrm{K}^{(i)}(t) = \frac{2\gamma^{(i)}\left(1 - \exp\left\{-\mathrm{D}^{(i)}t\right\}\right)}{\left(\mathrm{D}^{(i)} - \mathrm{c}^{(i)}\mathrm{a}^{(i)}\right) + \left(\mathrm{D}^{(i)} + \mathrm{c}^{(i)}\mathrm{a}^{(i)}\right)\exp\left\{-\mathrm{D}^{(i)}t\right\}},$$

$$\mathrm{J}^{(i)}(t) = \frac{2\mathrm{D}^{(i)}\exp\left\{-\frac{\left(\mathrm{D}^{(i)} + \mathrm{c}^{(i)}\mathrm{a}^{(i)}\right)}{2}t\right\}}{\left(\mathrm{D}^{(i)} - \mathrm{c}^{(i)}\mathrm{a}^{(i)}\right) + \left(\mathrm{D}^{(i)} + \mathrm{c}^{(i)}\mathrm{a}^{(i)}\right)\exp\left\{-\mathrm{D}^{(i)}t\right\}}.$$

The expression for a default free bond price can be obtained easily by substituting $\gamma^{(1)} = 0$ and $\gamma^{(2)} = 1$ in (3). Furthermore, if we set $\rho = 0$ in (3), we have the bond price expression under the celebrated Cox-ingersoll-Ross (1985) model in Cox et al. (1985). Due to the dependence of simultaneous event jumps of $Y^{(i)}$'s with sharing event jump frequency rate ρ, we have that

$$\mathbb{E}\left[\exp\left\{-\sum_{i=1}^{2}\Lambda^{(i)}(t)\right\}\right] \neq \mathbb{E}\left[\exp\left\{-\Lambda^{(1)}(t)\right\}\right]\mathbb{E}\left[\exp\left\{-\Lambda^{(2)}(t)\right\}\right].$$

However, if the event jump $Y^{(i)}$ for i = 1, 2 occurs by a Poisson process $M_t^{(i)}$ with its frequencyrate ρ respectively and everything else is independent of each other, the expression of the defaultable bond price is simply the product of the bond issuer's survival probability and the discount factor.

COPULA

The importance of introducing the correlation aspect between default intensity and interest rates was mentioned in n otes 8 of Kijima (2000) due to the strong evidence that the default intensity of corporate bonds varies with the business cycle. For that purpose, a copula function is an excellent candidate as it allows each variable being modelled to have different marginal distributions. Other than bond pricing, copulas have also been applied widely in capturing the dependence structure embedded in insurance portfolio as well as other financial products and indices (ignatieva & Platen 2010; Ma & Kim 2010; Mohd Ramli & Jang 2014; Shamiri et al. 2011).

The source of dependency between the variables $\lambda^{(1)}(t)$ and $\lambda^{(2)}(t)$ is the common event arrival process M_t, together with the dependence between the vector of jumps $(Y_j^{(1)}, Y_j^{(2)})$. We assume that the event arrival process M_t, i.e. the simultaneous jump process, follows a homogeneous Poisson process with frequency ρ and the vector of jumps is modelled using copulas. in other words, the joint distribution of the vector $(Y_j^{(1)}, Y_j^{(2)})$ is assumed to be of the form $C(F^{(1)},F^{(2)})$ with C being a given copula. The cumulative distribution function of the fg M, the g aussian and the Student-t copulas are given by the following in a consecutive manner:

$$C^{FGM}(u_1,u_2) = [1 + \theta(1 - u_1)(1 - u_2)]u_1u_2 \qquad (4)$$

$$C^{G}(u_1,u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)} \frac{\exp\left(-\frac{1}{2}\boldsymbol{\omega}^T\Theta^{-1}\boldsymbol{\omega}\right)}{2\pi\sqrt{|\Theta|}}du\,dx \qquad (5)$$

$$C_\nu^t(u_1,u_2) = \int_{-\infty}^{t_\nu^{-1}(u_1)}\int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^2|\Theta|}}\left(1 + \frac{\eta^T\Theta^{-1}\eta}{\nu}\right)du\,dx \qquad (6)$$

where $u_i \epsilon$ [0,1] for i = 1, 2, and the correlation parameter $\theta \epsilon$ [-1, 1]. For the g aussian and student-t copulas, the correlation parameter is contained in the correlation matrix $\Theta = \begin{bmatrix}1 & \theta\\ \theta & 1\end{bmatrix}$. We also define the vectors $\boldsymbol{\omega} = [\omega_1\ \omega_2]^T$ and $\boldsymbol{\eta} = [\eta_1\ \eta_2]^T$ where $\omega_i = \Phi^{-1}(u_i)$ and $\eta i = t_\nu^{-1}(u_i)$ are the inverse g aussian and inverse Student-t distribution with degrees of freedom , respectively, taken on the variables u_i. While any distribution can be considered for the marginal distributions of $Y_j^{(i)}$ in the vector of jumps $(Y_j^{(1)}, Y_j^{(2)})$, only the continuous marginals will ensure a uniquely defined copula distribution (see the Sklar's theorem in nelsen (2006), and Shamiri et al. (2011) for example). In this study, the jump size variables are assumed to follow the exponential marginal for simplicity of illustration.

The FGM copula is used in this study for its simplicity and analytical tractability which allows for the closed-form expressions to be easily derived. However, due to the weak dependence structure under the FGM copula, we propose to examine the bond price and yield under another two sided copulas, i.e. copula with dependence structure $\theta \in$ [-1, 1], which are the g aussian and the Student-t copulas. The g aussian copula was commonly used prior to the GFC 2008, while the Student-t copula is chosen as a potential candidate to represent the risky environment. In other words, the Student-t copula was chosen to incorporate the impact of higher frequency of concurring and opposing joint jump sizes, given its ability to capture variables with extreme values. Readers are referred to Mohd Ramli and Jang (2015) for the graphical illustrations of the three copulas when applied to two simulated processes. However, since the elliptical copulas do not admit analytical expression when combined with the chosen marginals, we evaluate the bond price numerically.

RESULTS AND DISCUSSION

Now we examine the behaviour of the defaultable zero coupon bond prices under three different copulas mentioned in Methods section. The hypothetical defaultable bond pays redemption value \$100 at maturity. f or simplicity, we assume that the jump sizes of both the bond issuer's default intensity (i = 1) and the market short rate (i = 2) are exponentially distributed. The defaultable bond price values P_t are computed using (7), and the bond yield d_t is obtained using the following formula:

$$d_t = \left(\frac{Future\ Value}{P_t}\right)^{\frac{1}{T-t}} - 1. \qquad (8)$$

We examine two scenarios whereby the exponential jump size parameters, $(\mu^{(1)}, \mu^{(2)})$ are assigned the values

($\mu_t^{(1)} = 100$, $\mu_t^{(2)} = 200$) and ($\mu_t^{(1)} = 5$, $\mu_t^{(2)} = 10$). The first set of parameters represents a safer environment due to low average jump sizes ($\frac{1}{100}$ and $\frac{1}{200}$ respectively), while the second set denotes a relatively riskier environment with relatively higher average jump sizes ($\frac{1}{5}$ and $\frac{1}{10}$, respectively). We assume an average of 4 jump occurrences per year (i.e. $\rho = 4$) and that the long term mean value for the issuer's default intensity and the market short rate = 0. Furthermore, to represent the relatively secure nature of a financial market relative to an issuer, we also assume that $c^{(1)} < c^{(2)}$ (i.e. the market short rate process reverts to its long term mean quicker than the default intensity process) and $\sigma^{(1)} > \sigma^{(2)}$ (i.e. default intensity process is more volatile than the market short rate process). The value of other parameters are summarized in Table 1. The degrees of freedom $\upsilon=3$ are used to compute the bond price under the Student-t copula with one-year maturity. The bond price values for maturity > 1 year can be found in the Appendix section.

TABLE 1. Parameter values of bond issuer's default intensity and short rate

	$c^{(i)}$	$\alpha^{(i)}$	$b^{(i)}$	$\sigma^{(i)}$	$\rho^{(i)}$	$\lambda_0^{(i)}$
Issuer (1)	3	-1	0	0.5	4	0.5
Short rate	0.5	-1	0	0.4	4	0.0025

Table 2 exhibits the bond price for each scenario with the corresponding yield in Table 3. The term 'Range' in Table 2 denotes the difference between the bond prices given by $\theta^{0.95}$ and $\theta^{-0.95}$, i.e. $\text{Price}^{\theta=-0.95} - \text{Price}^{\theta=+0.95}$.

TABLE 2. Zero coupon bond price under various copulas for t = 1

	$\mu_t^{(1)} = 100$ and $\mu_t^{(2)} = 200$			$\mu_t^{(1)} = 5$, $\mu_t^{(2)} = 10$		
θ	FGM	Gaussian	Student-t	FGM	Gaussian	Student-t
-0.95	92.627	92.529	89.424	57.797	57.359	51.019
-0.9	92.627	92.530	89.564	57.810	57.387	51.030
-0.5	92.627	92.626	89.578	57.910	57.669	51.275
0	92.628	92.628	90.002	58.036	58.036	51.867
0.5	92.629	92.631	90.030	58.163	58.470	52.791
0.9	92.629	92.633	90.121	58.264	58.866	53.883
0.95	92.629	92.634	90.151	58.277	58.870	54.065
Range	-0.2%	-0.105	-0.727	-0.48	-1.511	-3.046

We see that as θ progressed from negative to positive, the bond price figures in Table 2 demonstrate an increasing pattern while the bond yield figure in Table 3 shows a decreasing pattern under all copulas considered. In practice, the 'high yield, low price' pattern is usually seen when examining the price behaviour of a bond with high risk, implying that higher compensation is required to invest for a long period when the market is uncertain. In comparison with the other two copulas, the bond price values are the lowest under the Student-t copula, suggesting that a dependence structure under the Student-t copula could be a good candidate to depict a riskier environment.

TABLE 3. Zero coupon bond price under various copulas for t = 1

	$\mu_t^{(1)} = 100$ and $\mu_t^{(2)} = 200$			$\mu_t^{(1)} = 5$, $\mu_t^{(2)} = 10$		
θ	FGM	Gaussian	Student-t	FGM	Gaussian	Student-t
-0.95	7.960%	8.074%	11.827%	73.019%	74.340%	96.006%
-0.9	7.960%	8.074%	11.652%	72.982%	74.255%	95.963%

-0.5	7.960%	7.961%	11.635%	72.681%	73.405%	95.028%
0	7.959%	7.959%	11.109%	72.306%	72.306%	92.802%
0.5	7.958%	7.956%	11.074%	71.931%	71.029%	89.425%
0.9	7.957%	7.953%	10.962%	71.632%	69.880%	85.587%
0.95	7.957%	7.952%	10.925%	71.595%	69.866%	84.961%

Analogously, the bond yields are highest under the Student-t copula and lowest under the FGM copula. Additionally, the yield under Gaussian copula differs by less than ±0.15% than the FGM counterpart, when the environment is relatively safe. When comparing the bond yield across θ for both scenarios, we also noticed that the yields for the case of ($\mu_t^{(1)} = 5$, $\mu_t^{(2)} = 10$) are much higher than the yields given by the case of ($\mu_t^{(1)} =100$, $\mu_t^{(2)} =200$) for all copulas considered, i.e. by almost 10 times higher. This is not surprising as lower exponentially distributed jump size parameters indicate a higher average jump size, thereby indicating a relatively risky market environment.

It is also worth noting that the bond price and yield under the Student-t copula is not equal to the bond prices and yields as the Gaussian and FGM copula when θ = 0. In contrast to the general theorem of copula, the Student-t copula does not give an independence case when the dependence parameter θ = 0, and hence it would not result in product copula (Schmidt 2006).

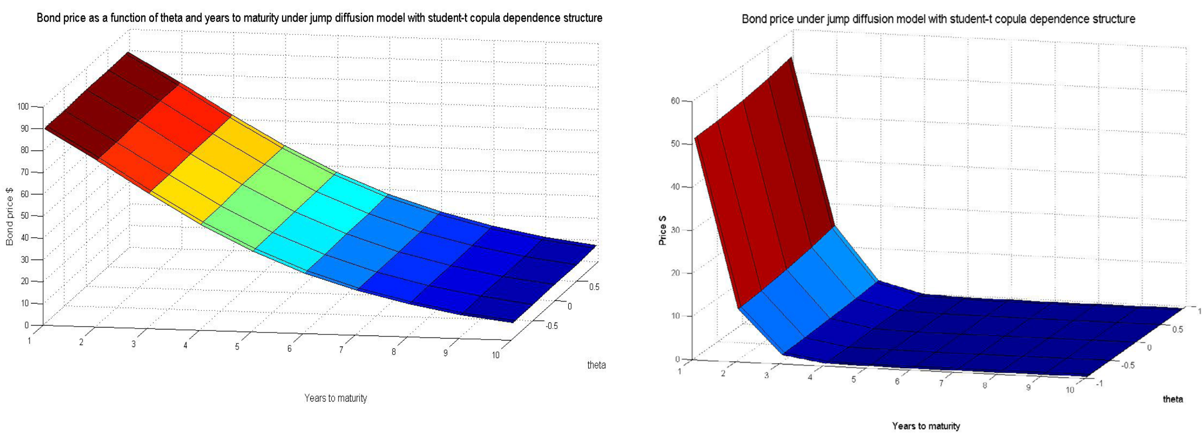

FIGURE 1. Bond price as a function of θ and maturity under the jump-diffusion model with Student-t copula dependence structure and jump sizes ($\mu_t^{(1)} = 100$, $\mu_t^{(2)} = 200$) (left) and ($\mu_t^{(1)} = 5$, $\mu_t^{(2)} = 10$) (right)

Figures 1 and 2 show the bond price and bond yield under the jump-diffusion process with the dependence structure captured by the Student-t copula as a function of maturity (T−t) (on the x-axis) and θ (on the y-axis). Under both scenarios of ($\mu_t^{(1)} = 100$, $\mu_t^{(2)} = 200$) and ($\mu_t^{(1)} = 5$, $\mu_t^{(2)} = 10$), the bond price decreased and yield increased as maturity increased reflecting the riskiness of the instrument in the long run which commands higher return for funds kept for a long period up to 10 years. Additionally, in the risky market environment i.e. when the ($\mu_t^{(1)} = 5$, $\mu_t^{(2)} = 10$), the bond is almost worthless as the term to maturity goes beyond 5 years. The bond price and bond yield under the Gaussian and the FGM copula show a similar pattern (see Appendix).

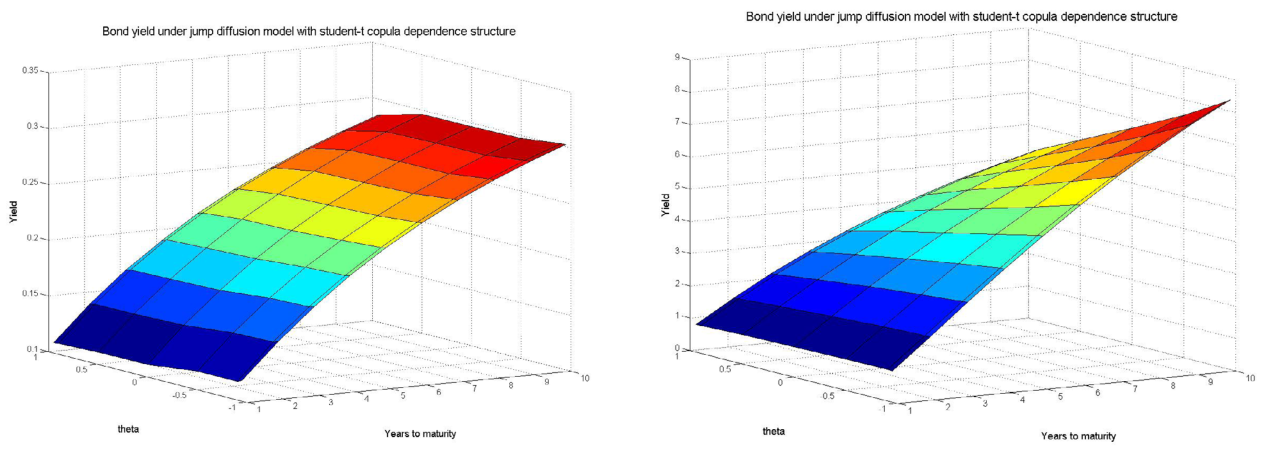

FIGURE 2. Bond yield as a function of θ and maturity under the jump-diffusion model with Student-t copula dependence structure and jump sizes ($\mu_t^{(1)} = 100$, $\mu_t^{(2)} = 200$) (left) and ($\mu_t^{(1)} = 5$, $\mu_t^{(2)} = 10$) (right)

CONCLUSION

This paper examined defaultable bond prices under the bivariate jump-diffusion model, whose jump sizes are dependent. The variables were the default intensity of a bond issuer and the short rate of interest, with exponentially distributed jump sizes and the dependence structure being captured by three copulas. The results indicated that the bond price under the Student-t copula showed the highest yield and had the widest range between both ends of the dependence parameters θ, suggesting that it is a better candidate to represent a risky environment relative to the other two copulas. We also found that in comparison to the relatively safe environment, the bond yield values under the risky environment are very high, i.e. above 100%, as the bond tenure increase up to 10 years, although both environments have an upward sloping term structure. It would be of interest to calibrate the model to the real bond data issued by a corporation to examine the dependency between its defaultability and short rate of interest, which we leave for further research.

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A1: BOND PRICE AND YIELD AS A FUNCTION OF TENOR AND θ WITH Y ~ Exp(100,200)

APPENDIX 1. Prices of zero coupon bond under jump diffusion model with student-t copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	89.424	75.583	61.109	47.708	36.232	26.924	19.668	14.172	10.103	7.134
-0.9	89.564	75.799	61.338	47.906	36.38	27.023	19.724	14.197	10.108	7.142
-0.5	89.578	75.880	61.524	48.201	36.758	27.443	20.149	14.599	10.446	7.391
0	90.002	76.521	62.185	48.758	37.161	27.693	20.273	14.6340	10.469	7.445
0.5	90.030	76.678	62.410	48.983	37.366	27.870	20.420	14.751	10.538	7.461
0.9	90.121	76.713	62.55	49.336	37.900	28.520	21.114	15.433	11.167	8.016
0.95	90.151	76.885	62.805	49.609	38.168	28.767	21.333	15.620	11.322	8.1418

APPENDIX 2. Prices of zero coupon bond under jump diffusion model with Gaussian copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	92.529	82.015	70.281	58.598	47.798	38.312	30.283	23.673	18.343	14.114
-0.9	92.530	82.016	70.284	58.602	47.803	38.318	30.290	23.680	18.350	14.121
-0.5	92.626	82.157	70.518	58.933	48.148	38.657	30.609	23.971	18.609	14.346
0	92.628	82.243	70.577	58.934	48.216	38.745	30.708	24.073	18.710	14.441
0.5	92.631	82.255	70.608	58.986	48.221	38.787	30.788	24.180	18.831	14.558
0.9	92.633	82.270	70.645	59.050	48.310	38.854	30.828	24.199	18.833	14.568
0.95	92.634	82.284	70.680	59.110	48.394	38.956	30.942	24.316	18.949	14.668

APPENDIX 3. Prices of zero coupon bond under jump diffusion model with FGM copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	92.627	82.332	70.805	59.332	48.714	39.358	31.400	24.804	19.442	15.146
-0.9	92.627	82.332	70.806	59.334	48.716	39.361	31.404	24.808	19.445	15.149
-0.5	92.627	82.336	70.814	59.349	48.736	39.386	31.431	24.837	19.474	15.177
0	92.628	82.340	70.824	59.366	48.762	39.417	31.466	24.873	19.510	15.211
0.5	92.629	82.344	70.835	59.384	48.787	39.448	31.501	24.909	19.546	15.246
0.9	92.629	82.347	70.843	59.400	48.807	39.473	31.528	24.938	19.575	15.273
0.95	92.629	82.347	70.844	59.401	48.810	39.476	31.532	24.942	19.579	15.279

APPENDIX 4. Yield (in %) of zero coupon bond under jump diffusion model with student-t copula dependence structure

θ	1	2	3	4	5	6	7	8	9	10
-0.95	11.827	15.024	17.842	20.324	22.513	24.445	26.152	27.665	29.008	30.217
-0.9	11.652	14.860	17.695	20.200	22.413	24.369	26.101	27.637	29.002	30.202
-0.5	11.635	14.799	17.576	20.015	22.160	24.050	25.717	27.192	28.530	29.756
0	11.109	14.317	17.158	19.671	21.894	23.863	25.607	27.154	28.499	29.662
0.5	11.074	14.199	17.017	19.533	21.760	23.731	25.477	27.027	28.406	29.634
0.9	10.962	14.173	16.930	19.319	21.415	23.256	24.879	26.312	27.580	28.707
0.95	10.925	14.046	16.771	19.155	21.244	23.079	24.695	26.122	27.386	28.507

APPENDIX 5. Yield (in %) of zero coupon bond under jump diffusion model with Gaussian copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	8.074	10.421	12.474	14.295	15.909	17.340	18.608	19.735	20.736	21.629
-0.9	8.074	10.421	12.473	14.294	15.907	17.337	18.604	19.730	20.731	21.623

-0.5	7.961	10.326	12.348	14.133	15.740	17.164	18.427	19.547	20.543	21.430
0	7.959	10.268	12.317	14.132	15.708	17.120	18.372	19.484	20.471	21.350
0.5	7.956	10.260	12.301	14.107	15.705	17.099	18.328	19.418	20.384	21.252
0.9	7.953	10.250	12.281	14.076	15.663	17.065	18.306	19.406	20.383	21.244
0.95	7.952	10.241	12.263	14.047	15.622	17.014	18.244	19.334	20.301	21.161

Appendix 6. Yield (in %) of zero coupon bond under jump diffusion model with FGM copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	7.960	10.209	12.197	13.940	15.470	16.814	17.996	19.038	19.958	20.773
-0.9	7.960	10.209	12.196	13.939	15.469	16.812	17.994	19.036	19.956	20.771
-0.5	7.960	10.206	12.192	13.933	15.459	16.800	17.979	19.018	19.936	20.749
0	7.959	10.204	12.186	13.924	15.448	16.785	17.961	18.996	19.911	20.721
0.5	7.958	10.201	12.181	13.915	15.436	16.770	17.942	18.975	19.887	20.694
0.9	7.957	10.197	12.176	13.908	15.426	16.757	17.927	18.957	19.867	20.672
0.95	7.957	10.198	12.176	13.908	15.425	16.756	17.922	18.955	19.865	20.670

A2: BOND PRICE AND YIELD AS A FUNCTION OF TENOR AND θ WITH Y ~ Exp(5,10)

APPENDIX 7. Prices of zero coupon bond under jump diffusion model with student-t copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	51.019	11.845	1.732	0.188	0.017	0.001	8.80E-05	5.6E-06	3.4E-07	2.0E-08
-0.9	51.030	11.893	1.751	0.192	0.017	0.001	9.4E-05	6.1E-06	3.7E-07	2.2E-08
-0.5	51.275	12.348	1.928	0.228	0.0224	0.002	0.0002	1.1E-05	7.6E-07	5.1E-08
0	51.867	13.089	2.204	0.287	0.0314	0.003	0.0003	2.3E-05	1.8E-06	1.4E-07
0.5	52.791	14.085	2.578	0.372	0.046	0.005	0.0005	4.9E-05	4.4E-06	3.9E-07
0.9	53.883	15.191	3.010	0.478	0.065	0.008	0.0009	9.9E-05	1.0E-05	1.0E-06
0.95	54.065	15.368	3.080	0.496	0.069	0.009	0.001	0.0001	1.1E-05	1.2E-06

Appendix 8. Prices of zero coupon bond under jump diffusion model with Gaussian copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	57.359	15.816	2.736	0.347	0.036	0.003	0.00024	1.7E-05	1.2E-06	7.6E-08
-0.9	57.387	15.862	2.757	0.352	0.036	0.003	0.0003	1.8E-05	1.3E-06	8.2E-08
-0.5	57.669	16.283	2.944	0.396	0.044	0.004	0.0004	2.8E-05	2.1E-06	1.5E-07
0	58.036	16.873	3.215	0.465	0.056	0.006	0.0006	4.9E-05	4.1E-06	3.3E-07
0.5	58.470	17.562	3.541	0.552	0.072	0.008	0.0009	8.7E-05	8.1E-06	7.4E-07

0.9	58.866	18.195	3.852	0.639	0.09	0.011	0.00130	0.00014	1.4E-05	1.4E-06
0.95	58.87	18.251	3.885	0.650	0.092	0.012	0.00136	0.00015	1.5E-05	1.6E-06

Appendix 9. Prices of zero coupon bond under jump diffusion model with FGM copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	57.797	16.488	3.036	0.419	0.048	0.005	0.0004	3.4E-05	2.7E-06	2.0E-07
-0.9	57.810	16.508	3.045	0.422	0.048	0.005	0.0004	3.5E-05	2.7E-06	2.1E-07
-0.5	57.910	16.669	3.120	0.440	0.051	0.005	0.0005	4.1E-05	3.3E-06	2.5E-07
0	58.036	16.873	3.215	0.465	0.056	0.006	0.001	4.9E-05	4.1E-06	3.3E-07
0.5	58.163	17.079	3.313	0.491	0.060	0.007	0.001	5.9E-05	5.1E-06	4.3E-07
0.9	58.264	17.246	3.394	0.512	0.065	0.007	0.001	6.9E-05	6.2E-06	5.3E-07
0.95	58.277	17.267	3.404	0.515	0.065	0.007	0.001	7.0E-05	6.3E-06	5.5E-07

Appendix 10. Yield (in %) of zero coupon bond under jump diffusion model with student-t copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	96.006	190.56	286.50	380.36	470.04	554.41	633.00	705.74	772.80	834.48
-0.9	95.963	189.97	285.05	377.81	466.23	549.27	626.50	697.90	763.65	824.08
-0.5	95.028	184.58	272.96	357.58	437.05	510.80	578.75	641.07	698.11	750.24
0	92.802	176.40	256.69	332.05	401.77	465.72	524.12	577.28	625.64	669.63
0.5	89.425	166.46	238.52	304.91	365.51	420.53	470.36	515.45	556.23	593.17
0.9	85.587	156.57	221.48	280.36	333.49	381.32	424.34	463.06	499.34	530.85
0.95	84.961	155.09	219.01	276.86	328.98	375.85	417.93	456.18	490.39	521.65

Appendix 14. Yield (in %) of zero coupon bond under jump diffusion model with Gaussian copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	74.340	151.45	231.86	312.16	390.11	464.40	534.33	599.63	660.27	716.40
-0.9	74.255	151.09	231.02	310.69	387.91	461.40	530.51	594.97	654.80	710.13
-0.5	73.405	147.82	223.88	298.54	370.03	437.40	500.23	558.45	612.17	661.62
0	72.306	143.45	214.50	282.99	347.64	407.88	463.55	514.75	561.72	604.73
0.5	71.029	138.62	204.52	266.92	325.00	378.54	427.61	472.44	513.32	550.59
0.9	69.880	134.44	196.11	253.64	306.59	354.99	399.04	439.07	475.41	508.42
0.95	69.866	134.08	195.26	252.22	304.58	352.36	395.83	435.29	471.11	503.63

Appendix 12. Yield (in %) of zero coupon bond under jump diffusion model with FGM copula dependence structure for years to maturity 1-10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	73.019	146.27	220.54	292.99	362.01	426.79	487.01	542.67	593.91	641.00
-0.9	72.982	146.12	220.22	292.46	361.24	425.78	485.75	541.17	592.18	639.05
-0.5	72.681	144.93	217.67	288.22	355.15	417.74	475.78	529.29	578.47	623.59
0	72.306	143.45	214.50	282.99	347.64	407.88	463.55	514.75	561.72	604.73
0.5	71.931	141.97	211.36	277.83	340.26	398.20	451.58	500.55	545.37	586.35
0.9	71.632	140.80	208.88	273.75	334.44	390.59	442.19	489.43	532.59	572.00
0.95	71.595	140.65	208.57	273.25	333.72	389.64	441.02	488.05	531.01	570.23