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Using Hybrid of Block-Pulse Functions and Bernoulli Polynomials to Solve Fractional Fredholm-Volterra Integro-Differential Equations

(Menggunakan Fungsi Blok-Denyut Hibrid dan Polinomial Bernoulli untuk Menyelesaikan Persamaan Pembezaan-Integro Fredholm-Volterra Pecahan)

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ABSTRCT

Fractional integro-differential equations have been the subject of significant interest in science and engineering problems. This paper deals with the numerical solution of classes of fractional Fredholm-Volterra integro-differential equations. The fractional derivative is described in the Caputo sense. We consider a hybrid of block-pulse functions and Bernoulli polynomials to approximate functions. The fractional integral operator for these hybrid functions together with the Legendre-Gauss quadrature is used to reduce the computation of the solution of the problem to a system of algebraic equations. Several examples are given to show the validity and applicability of the proposed computational procedure.

Keywords: Bernoulli polynomials; Block-pulse functions; fractional integro-differential equations; hybrid functions; Caputo derivative

ABSTRACK

Persamaan pembezaan integro pecahan telah menjadi subjek penting dalam masalah sains dan kejuruteraan. Makalah ini berkaitan dengan penyelesaian berangka kelas persamaan pembezaan integro Fredholm-Volterra pecahan. Terbitan pecahan diterangkan dalam maksud Caputo. Fungsi hibrid blok-denyutan dan polinomial Bernoulli dipertimbangkan untuk penghampiran fungsi. Pengoperasi kamiran pecahan untuk fungsi hibrid bersama-sama dengan kuadratur Legendre-Gauss digunakan untuk mengurangkan pengiraan penyelesaian masalah kepada sistem persamaan algebra. Beberapa contoh diberikan untuk menunjukkan kesahihan dan kebolehgunaan prosedur pengiraan yang dicadangkan.

Kata kunci: Fungsi blok-denyutan; fungsi hibrid; persamaan pembezaan integro pecahan; polinomial Bernoulli; terbitan Caputo

INTRODUCTION

The study of the fractional derivative has its origins in 1695. Leibniz and L'Hospital can be considered as the first two mathematics to discuss on fractional derivative (Kilbas et al. 2006). In recent years, the study of fractional differential equations and fractional integro-differential equations (FIDEs) has gained high interest because of its considerable application in science and engineering (Abdullah 2013; Abuasad & Hashim 2018; Dascioglu & Bayram 2019; Kilbas et al. 2006; Miller & Ross 1993; Podlubny 1999; Saadatmandi & Dehghan 2011a, 2011b). FIDEs occur in many physical processes such as chemistry, electromagnetism, acoustics, and viscoelastic materials (Kilbas et al. 2006; Miller & Ross 1993; Podlubny 1999). Unfortunately, analytic solutions of the most FIDEs cannot be obtained explicitly, therefore, numerical techniques must be used. In the past two decades or so, considerable numerical methods to solve FIDEs have been given such as collocation method (Rawashdeh 2006; Saadatmandi & Dehghan 2011a; Saadatmandi et al. 2018), Adomian's decomposition method (Mittal & Nigam 2008; Momani & Noor 2006), variational iteration method (Kurulay & Secer 2011), fractional differential transform method (Arikoglu & Ozkol 2009), modification of hat functions (Nemati & Lima 2018), hybrid functions (Mashayekhi & Razzaghi 2015), Taylor expansion method (Huang et al. 2011), Sinc-collocation method (Bayram et al. 2018) and the wavelet method (Meng et al. 2015; Saeedi et al. 2011; Zhu & Fan 2012).

In this paper, we study the numerical solution of following fractional Fredholm-Volterra integro-differential equation (Rahimkhani et al. 2017)

$$D^{v} y(x) = \lambda y(x) + g(x) + \lambda_1 \int_0^x k_1(x,t)\,[y(t)]^{q1}$$

$$[D^{v_1} y(t)]^{q_2}\, dt + \lambda_2 \int_0^1 k_2(x,t) [y(t)]^{q_3}$$

$$[D^{v_2} y(t)]^{q_4}\, dt, \qquad (1)$$

$$0 \le x,\ t \le 1,\ n - 1 < v \le n,\ 0 \le v_1 v_2 \le v.$$

with initial conditions

$$y^{(k)}(0) = \mu_k, \quad k = 0, 1, \ldots, n - 1. \qquad (2)$$

Wherever, the mentioned functions g, k_1 and k_2 are real-valued functions. Also, $k1$, and k_2 are assumed to be separable kernels. In addition, λ, λ_1, λ_2, μ_k, $k = 0, 1, \ldots, n - 1$ are real given numbers and q_i, $i = 1, \ldots, 4$, are positive integers. Here, D^{v}, D^{v_1} and D^{v_2} are the Caputo type fractional derivatives of order v, v_1 and v_2 respectively.

The literature of numerical methods contains little on the solutions of (1). Rahimkhani et al. (2017) used fractional-order Bernoulli functions for solving (1). Also, for the case of $q_2 = q_4 = 0$, Keshavarz et al. (2019) applied Bernoulli wavelets to approximate solutions of (1). Moreover, Meng et al. (2015) used the Legendre wavelets method to solve (1) with $\lambda = q_2 = q_4 = 0$ and $q_1 = q_3 = 1$.

The main idea of this work is to apply a hybrid of block-pulse functions and Bernoulli polynomials together with the collocation method to solve the problem (1)-(2). The main advantage behind this approach is that it reduces the solution of fractional Fredholm-Volterra integro-differential (1) to those of solving a system of algebraic equations thus seriously simplifying the problem. Numerical methods based on the hybrid functions of block-pulse and Bernoulli polynomials are a nice and powerful approach for the numerical solution of problems arising from engineering applications including optimal control of delay systems (Haddadi et al. 2012), nonlinear constrained optimal control problems (Mashayekhi et al. 2012), fractional Volterra integro-differential equations (Mashayekhi & Razzaghi 2015), multi-delay systems (Mashayekhi et al. 2016) and fractional Bagley-Torvik equation (Mashayekhi & Razzaghi 2016).

This paper is organized in the following way: in the next section, some basic results of fractional calculus and some properties of the hybrid of block-pulse functions and Bernoulli polynomials required for our subsequent development are given. In subsequent section, the new method proposed in this paper is presented. In the following section, some numerical results are given to clarify the method. Last section contains a brief conclusion.

Preliminaries and Notations

THE FRACTIONAL DERIVATIVE AND INTEGRAL

Definition 2.1 The Riemann-Liouville fractional integral of order α is defined as (Podlubny 1999)

$$I^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s)\,ds, & \alpha > 0, \\ f(t), & \alpha = 0, \end{cases} \qquad (3)$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.2 The Caputo fractional derivative with order $\alpha > 0$ is defined by (Podlubny 1999)

$$D^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n, \; n \in \mathbb{N}, \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}. \end{cases} \qquad (4)$$

We have the following properties (Podlubny 1999):

1. $D^{\alpha} C = 0$ (C is a constant).

2. $D^{\alpha} t^m = \begin{cases} 0, & m < \lceil \alpha \rceil, \\ \dfrac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} t^{m-\alpha}, & m \geq \lceil \alpha \rceil. \end{cases}$

where $\lceil . \rceil$ is the ceiling function and $m \in \mathbb{N}$

3. D^{α} and I^{α} are linear operators.

4. $D^{\alpha} I^{\alpha} f(t) = f(t)$.

5. $I^{\alpha} (D^{\alpha} f(t)) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \dfrac{t^k}{k!}$,

$n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$.

6. $D^{\alpha} f(t) = I^{m-\alpha} D^{m} f(t)$, $m \in \mathbb{N}$

HYBRID OF BLOCK-PULSE FUNCTIONS AND BERNOULLI POLYNOMIALS

The well-known Bernoulli polynomials of the order m are defined in the interval [0, 1] with the following formula (Costabile et al. 2006)

$$\beta_m(t) = \sum_{k=0}^{m} \binom{m}{k} \alpha_k t^{m-k}, \qquad (5)$$

where Bernoulli numbers α_k, can be defined by Costabile et al. (2006)

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}. \qquad (6)$$

The Bernoulli numbers α_k are rational numbers and the first few are

$$\alpha_0 = 1, \; , \alpha_1 = \frac{-1}{2}, \alpha_2 = \frac{1}{6}, \alpha_4 = \frac{-1}{30}, \alpha_6 = \frac{1}{42}.$$

It can be shown that $\alpha_{2k+1} = 0$ for $k \geq 1$. Now, hybrid functions $b_{nm}(t)$, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M$, are defined on the interval [0, 1] as (Mashayekhi et al. 2012)

$$b_{nm}(t)=\begin{cases}\beta_m(Nt-n+1), & t\in[\frac{n-1}{N},\frac{n}{N}).\\ 0, & \text{otherwise.}\end{cases} \quad (7)$$

Here, m and n are the order of Bernoulli polynomials and block-pulse functions, respectively.

Function approximation and error estimation

Let $H = L^2[0,1]$. . We define the approximation space as

$$Y = \text{span }\{b_{10}(t), b_{20}(t), ..., b_{N0}(t), b_{11}(t), b_{21}(t), ..., b_{N1}(t)...,$$
$$b_{1M}(t), ..., b_{NM}(t)\}.$$

Clearly, Y is a finite-dimensional subspace of H, thus any function $f\in H$ has a best unique approximation out of Y (Mashayekhi et al. 2012). That is, given $f\in H$ there exists $P_M^N f\in Y$ such that

$$\| f - P_M^N f\| \le \| f - y\|,$$

for all $y\in Y$. Therefore, there exist unique coefficients $c_{10}, c_{20}, ..., c_{NM}$ such that

$$f\simeq P_M^N f=\sum_{m=0}^{M}\sum_{n=1}^{N} c_{nm}b_{nm}(t)=C^T B(t), \quad (8)$$

where

$$C=[c_{10},...,c_{N0},c_{11},...,c_{N1},...,c_{1M},...,c_{NM}]^T, \quad (9)$$

and

$$B(t)=[b_{10}(t),...,b_{N0}(t),b_{11}(t),...,b_{N1}(t),...,b_{1M}(t),...,$$
$$b_{NM}(t)]^T. \quad (10)$$

The following error bound was proved in (Mashayekhi et al. 2016).

Theorem 2.3 Suppose $f\in H^{\mu}[0, 1]$ with μ 0, and $M \geq \mu - 1$, then

$$\| f - P_M^N f \|_{L^2[0,1]} \le cM^{-\mu}N^{-\mu}\| f^{(\mu)}\|_{L^2[0,1]}$$

and for $r \geq 1$,

$$\| f - P_M^N f\|_{H^r[0,1]} \le cM^{2r-\frac{1}{2}-\mu}N^{r-\mu}\ \| f^{(\mu}\|_{L^2[0,1]},$$

where c depends μ on .

Riemann-Liouville fractional integral operator for $B(t)$
The fractional integration of the $B(t)$, defined in (10), is given by

$$I\ \ B(t)=\overline{B}(t,\ \ ), \quad (11)$$

where

$$\overline{B}(t,\alpha)=[I^{\alpha}b_{10}(t),...,I^{\alpha}b_{N0}(t),I^{\alpha}b_{11}(t),...,I^{\alpha}b_{N1}(t),$$
$$...,I^{\alpha}b_{1M}(t),I^{\alpha}b_{2M}(t),...,I^{\alpha}b_{NM}(t)]^T, \quad (12)$$

and $I^{\alpha}b_{nm}(t)$ is obtained from (Mashayekhi & Razzaghi 2016, 2015)

$$I^{\alpha}b_{nm}(t)=\begin{cases}0, & t\in(-\infty,\frac{n-1}{N}),\\ (t-\frac{n-1}{N})^{\alpha}d_{nm}(t), & t\in[\frac{n-1}{N},\frac{n}{N}),\\ (t-\frac{n-1}{N})^{\alpha}d_{nm}(t)-(-1)^m(t-\frac{n}{N})^{\alpha}\overline{d}_{nm}(t), & t\in[\frac{n}{N},\infty),\end{cases} \quad (13)$$

where

$$d_{nm}(t)=\sum_{k=0}^{m}\binom{m}{k}a_{m-k}N^k\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}(t-\frac{n-1}{N})^k,$$

$$\overline{d}_{nm}(t)=\sum_{k=0}^{m}(-1)^k\binom{m}{k}a_{m-k}N^k\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}(t-\frac{n}{N})^k.$$

LEGENDRE-GAUSS QUADRATURE

Suppose that $\{\tau_q\}_{q=0}^{n}$ are the distinct roots of the Legendre polynomial of degree $n + 1$. The $(n + 1)$ -point Legendre-Gauss quadrature rule for approximating integral of a function $g(t)$ over the interval (a, b) , is given by (Canuto et al. 1988)

$$\int_a^b g(t)dt\simeq\frac{b-a}{2}\sum_{q=0}^{n}\omega_q g(\sigma_q), \quad (14)$$

where

$$\sigma_q=\frac{b-a}{2}\tau_q+\frac{b+a}{2},\ \omega_q=\frac{2}{(1-\tau_q^2)(L_{n+1}'(\tau_q))^2},$$
$$q=0,...,n. \quad (15)$$

Here, $\{\sigma_q\}_{q=0}^{n}$ and $\{w_q\}_{q=0}^{n}$ are Legendre-Gauss quadrature nodes and weights, respectively. This quadrature formula is exact for all $g(x)$ which are polynomials of at most degree $2n + 1$.

SOLUTION OF THE PROBLEM (1)-(2)

In this part, we apply hybrid of block-pulse functions and Bernoulli polynomials to approximate solutions of fractional Fredholm-Volterra integro-differential equation (1)-(2). For this purpose, we let

$$D^{v} y(x) = C^{T} B(x). \tag{16}$$

Thanks to (16) and in the presence of some properties of fractional integration, we have

$$y(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} y^{(k)}(0) = I^{v} D^{v} y(x) = C^{T} I^{v} B(x). \tag{17}$$

Employing (2), (11) and (17) we get

$$y(x) = C^{T} \overline{B}(x, v) + \sum_{k=0}^{n-1} \frac{x^k}{k!} \mu_k, \tag{18}$$

also, using (17) we obtain

$$\begin{aligned} D^{v_i} y(x) &= D^{v_i} \left(\sum_{k=0}^{n-1} \frac{x^k}{k!} \mu_k \right) + C^{T} D^{v_i} I^{v} B(x) \\ &= \sum_{k=0}^{n-1} \frac{D^{v_i}(x^k)}{k!} \mu_k + C^{T} I^{v-v_i} B(x) \\ &= \sum_{k=\lceil v_i \rceil}^{n-1} \frac{t^{k-v_i} \mu_k}{\Gamma(k+1-v_i)} + C^{T} \overline{B}(x, v-v_i), \quad i = 1, 2. \end{aligned} \tag{19}$$

Substituting (16), (18) and (19) into (1) gives

$$\begin{aligned} & C^{T} B(x) - \lambda (C^{T} \overline{B}(x,v) + \sum_{k=0}^{n-1} \frac{x^k}{k!} \mu_k) - g(x) \\ & -\lambda_1 \int_0^x k_1(x,t) [C^{T} \overline{B}(t,v) + \sum_{k=0}^{n-1} \frac{t^k}{k!} \mu_k]^{q_1} [C^{T} \overline{B}(t, v-v_1) + \\ & \sum_{k=\lceil v_1 \rceil}^{n-1} \frac{t^{k-v_1} \mu_k}{\Gamma(k+1-v_1)}]^{q_2} dt \\ & -\lambda_2 \int_0^1 k_2(x,t) [C^{T} \overline{B}(t,v) + \sum_{k=0}^{n-1} \frac{t^k}{k!} \mu_k]^{q_3} [C^{T} \overline{B}(t, v-v_2) \\ & \sum_{k=\lceil v_2 \rceil}^{n-1} \frac{t^{k-v_2} \mu_k}{\Gamma(k+1-v_2)}]^{q_4} dt = 0. \end{aligned} \tag{20}$$

Now, by collocating (20) at $N(M+1)$ points $x = x_i$ and by using Legendre-Gauss quadrature rule (14), we get

$$\begin{aligned} & C^{T} B(x_i) - \lambda (C^{T} \overline{B}(x_i, v) + \sum_{k=0}^{n-1} \frac{x_i^k}{k!} \mu_k) - g(x_i) \\ & -\lambda_1 \sum_{j=0}^{n} \frac{x_i}{2} \omega_j k_1(x_i, z_{ij}) \left[C^{T} \overline{B}(z_{ij}, v) + \sum_{k=0}^{n-1} \frac{(z_{ij})^k}{k!} \mu_k \right]^{q_1} \\ & \times \left[C^{T} \overline{B}(z_{ij}, v-v_1) + \sum_{k=\lceil v_1 \rceil}^{n-1} \frac{(z_{ij})^{k-v_1} \mu_k}{\Gamma(k+1-v_1)} \right]^{q_2} \\ & -\lambda_2 \sum_{j=0}^{n} \frac{1}{2} \omega_j k_2(x_i, \overline{z}_j) \left[C^{T} \overline{B}(\overline{z}_j, v) + \sum_{k=0}^{n-1} \frac{(\overline{z}_j)^k}{k!} \mu_k \right]^{q_3} \\ & \times \left[C^{T} \overline{B}(\overline{z}_j, v-v_2) + \sum_{k=\lceil v_2 \rceil}^{n-1} \frac{(\overline{z}_j)^{k-v_2} \mu_k}{\Gamma(k+1-v_2)} \right]^{q_4} \\ & = 0, \quad i = 0, 1, \ldots, N(M+1) - 1. \end{aligned} \tag{21}$$

Here, we use uniform collocation points

$$x_i = \frac{2i+1}{2N(M+1)}, \quad i = 0, 1, \ldots, N(M+1) - 1. \tag{22}$$

Also, in (21), $z_{ij} = \frac{x_i \tau_j + x_i}{2}$, $\overline{z}_j = \frac{\tau_j + 1}{2}$, moreover ω_j and τ_j are Legendre-Gauss weights and nodes, respectively. Therefore, (21) generate a set of N $(M + 1)$ nonlinear algebraic equations, which can be solved for the unknown vector C. Throughout this paper, we use Maple's *fsolve* command to find the unknown vector C from this system of algebraic equations.

NUMERICAL EXAMPLES

In this section, we present some examples to illustrate the efficiency and validity of our method for solving the problem (1)-(2). We implemented our method in a personal computer with 3.40 GHz Intel Core 7. Also, we use 5-point Legendre-Gauss quadrature rule.

Example 4.1 Let us first consider the nonlinear fractional Volterra integro-differential equation (Ghazanfari et al. 2010)

$$D^{v} y(x) = 1 + \int_0^x y(t) D^{v} y(t) dt, \quad 0 \le x \le 1, \quad 0 \le x \le 1, \tag{23}$$

with the initial condition $y(0) = 0$.

The exact solution, when $v = 1$, is $y(x) = \sqrt{2} \tan\left(\frac{\sqrt{2}}{2} x\right)$.

To solve this problem by the present method, let $D^{v}y(x) = C^{T} B(x)$. Also, by using (18), we get $y(x) = C^{T}B(x, v)$. Employing (21), we obtain

$$C^{T} B(x_i) - 1 - \sum_{j=0}^{n} \frac{x_i}{2} \omega_j C^{T} \overline{B}(z_{ij}, v) C^{T} B(z_{ij}) = 0,$$

$$i = 0, 1, \ldots, N(M+1) - 1. \tag{24}$$

Finally, we have N $(M + 1)$ algebraic equations. By solving these equations, the unknown vector C is obtained. The numerical results for y(x) with $N = 1$, $M = 5$ and $v = 0.7, 0.8, 0.9, 1$ are plotted in Figure 1. From Figure 1, we see that as v approaches 1, the approximate solutions converge to the exact solution. i.e. in the limit, the solution of the nonlinear fractional Volterra integro-differential equation approaches to that of the integer-order Volterra integro-differential equation. Also, in Figure 2 the logarithmic graphs of absolute error functions are plotted for $v = 1$ and for different values of M and N. This figure illustrate that the errors decay as M and N increases. Moreover, in Table 1, absolute errors and CPU times (in seconds) by choosing $v = 1$, $m = 5$ together with $N = 2, 3, 4$ are reported. In this table, we also compare our method with $M = 5$ and different values of N together with the results obtained with $M = 5$ by using the fractional-order Bernoulli functions given in Rahimkhani et al. (2017).

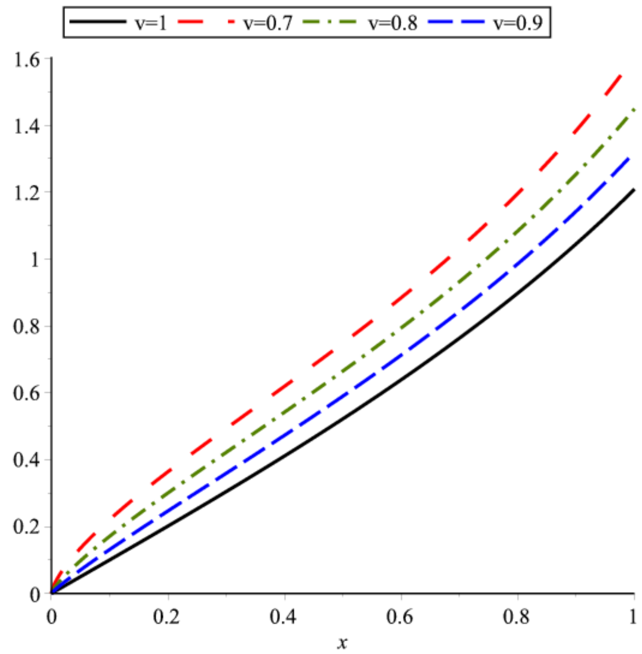


FIGURE 1. Comparison of $y(x)$ for $N = 1$, $M = 5$and with v = 0.7, 0.8, 0.8, 1 for Example 4.1

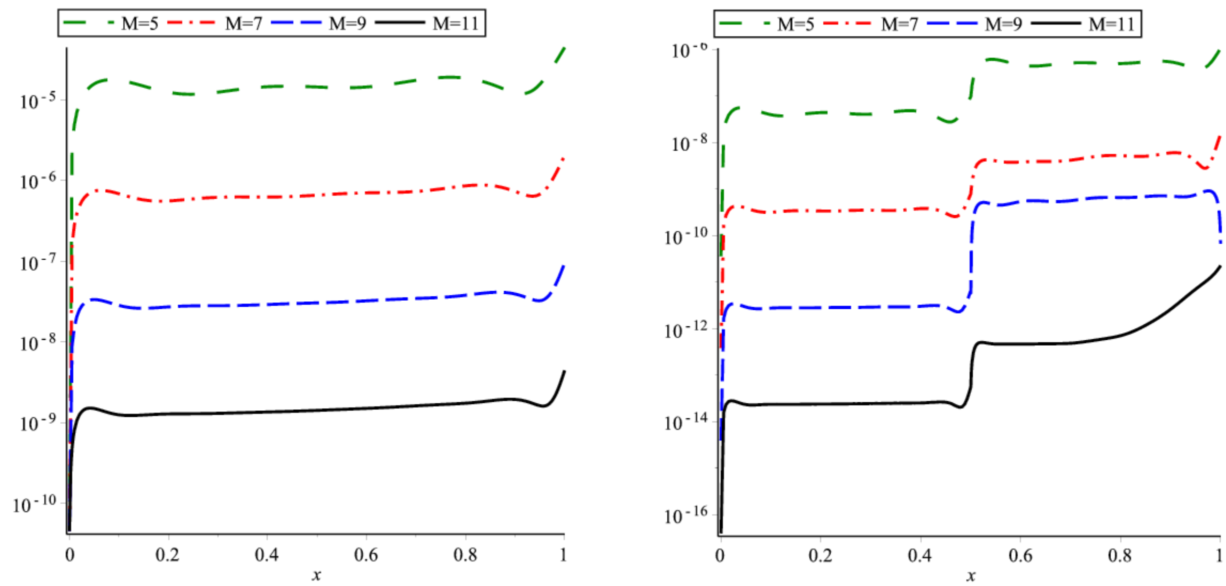


FIGURE 2. Graph of absolute error function for different values of M and with $N = 1$ (left) and $N = 2$ (right) for Example 4.1

TABLE 1. The comparison of absolute errors between the present method and the result given in (Rahimkhani et al. 2017) for Example 4.1

x	Method of (Rahimkhani et al. 2017) ($M = 5$)	Present method ($M = 5$)		
		$N = 2$	$N = 3$	$N = 4$
0.2	5.23 x 10-6	4.43 x 10-8	1.94 x 10-9	2.65 x 10-10
0.4	1.18 x 10-6	4.65 x 10-8	1.07 x 10-8	1.04 x 10-9
0.6	1.00 x 10-5	5.00 x 10-7	1.19 x 10-8	3.30 x 10-9
0.8	1.41 x 10-5	5.25 x 10-5	5.54 x 10-8	1.20 x 10-8
CPU time(s)	Not reported			

Example 4.2 Consider the nonlinear fractional Fredholm integro-differential equation given in Zhu dan Fan (2012) by

$$D^v y(x) = 1 - \frac{x}{4} + \int_0^1 xt[y(t)]^2 dt, \qquad 0 \le x < 1,$$

$$0 \le x < 1, \quad 0 < v \le 1, \tag{25}$$

with the initial condition $y(0) = 0$

The exact solution, when $v = 1$, , is $y(x) = x$. According to (21), we obtain

$$C^T B(x_i) - 1 + \frac{x_i}{4} - \sum_{j=0}^{n} \frac{x_i}{2} \omega_j \bar{z}_j \left[C^T \bar{B}\left(\bar{z}_j, v\right)\right]^2 = 0, \; i = 0,1,...,$$

$$N(M+1) - 1, \tag{26}$$

which is a nonlinear system of algebraic equations. By solving this system we can obtain the unknown vector C. The numerical results for $N = 1$, $M = 5$ and for different values of v are shown in Figure 3. Once again, from Figure 3, we see that as $v \to 1$, the approximate solutions converges to the exact solution. Not that, for $v = 1$, By choosing, $M = N = 1$ we get the exact solution. However, with the technique presented in Zhu and Fan (2012), the exact solution cannot be obtained. In addition, for $v = 1$, the *CPU* time of our method is 0.904 second.

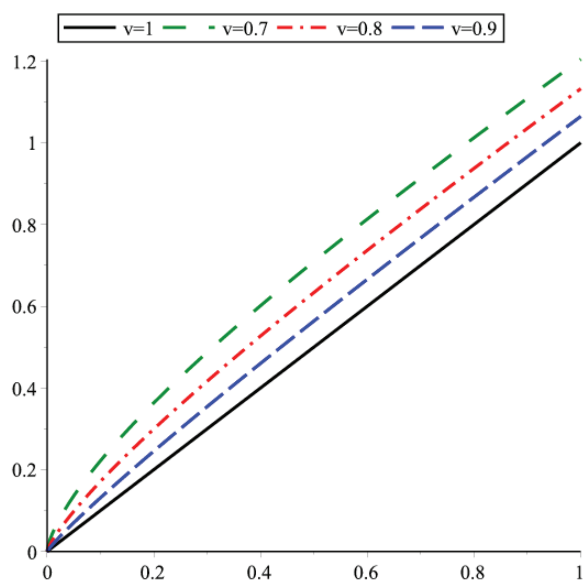


FIGURE 3. Comparison of $y(x)$ for $N = 1$, $M = 5$ and with $v = 0.7, 0.8, 0.9, 1$ for Example 4.2

Example 4.3 Consider the following fractional mixed Fredholm-Volterra integro-differential equation (Meng et al. 2015)

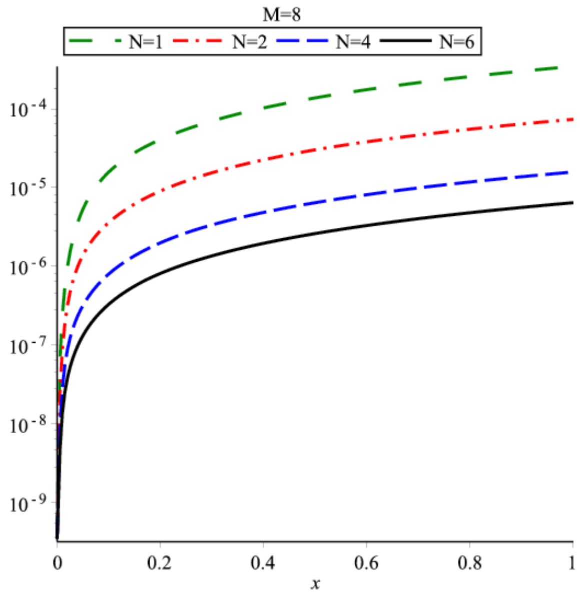
$$D^{2.3}y(x) = g(x) + \frac{1}{4}\int_0^x (x-t)y(t)dt + \frac{1}{2}\int_0^1 xty(t)dt,$$
$$0 \le x \le 1, \tag{27}$$

with the initial conditions $y(0) = y'(0) = y''(0) = 0$, and $g(x) = \frac{\Gamma(4.5)}{\Gamma(2.2)}x^{1.2} - \frac{1}{99}x^{5.5} - \frac{1}{11}x.$

The exact solution is $y(x) = x^2$. For this problem, (21), generate the following linear system of algebraic equations:

$$C^T B(x_i) - g(x_i) - \frac{1}{8}\sum_{j=0}^{n} x_i\omega_j(x_i - z_{ij})C^T\overline{B}(z_{ij}, 2.3) - \frac{1}{4}$$
$$\sum_{j=0}^{n} x_i\omega_j \overline{z}_j C^T \overline{B}(\overline{z}_j, 2.3) = 0, \quad i = 0,1,\dots,N(M+1)-1. \tag{28}$$

In Figure 4, the logarithmic graphs of absolute error functions are plotted for various values of M and N. From this figure, we can find that the errors decay as M and N increases. Moreover, for the purpose of comparison, in Table 2 we compare the absolute error of our method with $N = 2$ and different values of M together with the results obtained by using Adomian decomposition method (ADM) and the Legendre wavelets method (LWM) given in Meng et al. (2015). In addition, in Table 2, the CPU times of our method are reported.



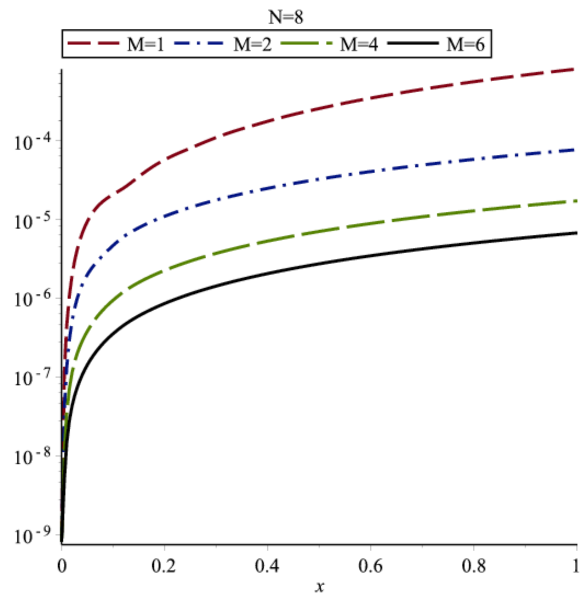


FIGURE 4. Graph of absolute error functions for various values of M and N for Example 4.3

TABLE 2. The comparison of absolute errors at some points for the present method and the result given in (Meng et al. (2015) for Example 4.3

x	LWM			Present method	
	$m = 2, k = 4$	$m = 2, k = 5$	$n = 5$	$M = 6, N = 2$	$M = 7, N = 2$
1/8	4.87 x 10^{-5}	6.64 x 10^{-5}	1.02 x 10^{-5}	9.89 x 10^{-6}	6.76 x 10^{-6}
2/8	8.92 x 10^{-5}	4.53 x 10^{-5}	4.20 x 10^{-5}	2.47 x 10^{-5}	1.69 x 10^{-5}
3/8	7.09 x 10^{-5}	3.14 x 10^{-5}	9.24 x 10^{-5}	4.21 x 10^{-5}	2.89 x 10^{-5}
4/8	2.36 x 10^{-4}	7.37 x 10^{-5}	4.17 x 10^{-4}	6.15 x 10^{-5}	4.20 x 10^{-5}
5/8	7.11 x 10^{-4}	2.44 x 10^{-4}	8.16 x 10^{-4}	8.16 x 10^{-5}	5.70 x 10^{-5}
6/8	2.51 x 10^{-3}	3.81 x 10^{-4}	2.31 x 10^{-3}	1.03 x 10^{-4}	7.30 x 10^{-5}
7/8	3.04 x 10^{-3}	6.02 x 10^{-4}	8.07 x 10^{-3}	1.25 x 10^{-4}	9.02 x 10^{-5}
CPU time(s)	Not reported	Not reported	Not reported	0.530	0.546

Example 4.4 Consider the following nonlinear fractional mixed Fredholm-Volterra integro-differential equation (Meng et al. 2015)

$$D^{2.2}y(x) = g(x) + \frac{1}{3}\int_0^x (x+t)[y(t)]^2 dt + \frac{1}{4}\int_0^1$$

$$(x-t)[y(t)]^3 dt, 0 \le x \le 1, \qquad (29)$$

with the initial conditions $y(0) = y'(0) = y''(0) = 0$ and

$$g(x) = \frac{\Gamma(4)}{\Gamma(1.8)}x^{0.8} - \frac{5}{56}x^8 - \frac{1}{40}x + \frac{1}{44}.$$

The exact solutin of this problem is $y(x) = x^3$. Here, (21), generate the following $m = N(M + 1)$ nonlinear system of algebraic equations:

$$C^T B(x_i) - g(x_i) - \frac{1}{6}\sum_{j=0}^{n} x_i \omega_j (x_i + z_{ij})\left[C^T \overline{B}(z_{ij}, 2.2)\right]$$
$$-\frac{1}{8}\sum_{j=0}^{n}(x_i - \overline{z}_j)\omega_j\left[C^T \overline{B}(\overline{z}_j, 2.2)\right]^3$$
$$= 0, \quad i = 0,1,...,N(M+1)-1. \qquad (30)$$

For the purpose of comparison of our method with the Legendre wavelets method (LWM) and CAS wavelet method (CASW) given in (Meng et al. 2015) we define l_∞ norm of absolute errors by

$$\| e_m \|_\infty = \max_{1 \le i \le m}\{| y(x_i) - y_m(x_i) |\},$$

where y is the exact solution; and y_m is the approximate solution. The results are summarized in Table 3. Also, in Table 3, the CPU times of our method are reported. According to Table 3, we find that the presented method provides accurate results.

TABLE 3. A comparison of $\| e_m \|_\infty$ between the present method and the result given in Meng et al. (2015) for Example 4.4

	LWM		CASW		Present method	
	$m = 16$	$m = 32$	$m = 16$	$m = 32$	$m = 9$ ($M = 8, N = 1$)	$m = 20$ ($M = 9, N = 2$)
$\| e_m \|_\infty$	4.43 x 10-3	2.14 x 10-3	7.84 x 10-2	5.24 x 10-3	7.85 x 10-4	1.89 x 10-4
CPU Time(s)	0.823	2.427	Not reported		0.515	2.434

As an another approach to this problem, we choose $N = 1$ and $M = 0$. Let

$$D^3 y(x) = C^T B(x) = c_{10} b_{10}(x) \quad ,$$

thus similar to (17) we get

$$y(x) = C^T \overline{B}(x,3) = c_{10} I^3 b_{10}(x). \qquad (31)$$

Also, by using (31), we obtain

$$D^{2.2} y(x) = C^T \overline{B}(x,0.8) = c_{10} I^{0.8} b_{10}(x). \qquad (32)$$

Substituting (31) and (32) in (29) and applying the Legendre-Gauss quadrature, we have

$$c_{10} I^{0.8} b_{10}(x_0) - g(x_0) - \frac{1}{6}\sum_{j=0}^{n} x_0 \omega_j \left(x_0 + z_{0j}\right)\left[c_{10} I^3 b_{10} (z_{0j})\right]^2 - \frac{1}{8}\sum_{j=0}^{n} \omega_j \left(x_0 - \bar{z}_j\right)\left[c_{10} I^3 b_{10}\left(\bar{z}_j\right)\right]^3 = 0, \qquad (33)$$

where $x_0 = 1/2$ is a collocation point. By solving this equation we get $C_{10} = 6$. Thus, using (31), we get $y(x) = x^3$, which is the exact solution of the problem.

Conclusion

In this paper, a collocation method with a hybrid of block-pulse functions and Bernoulli polynomials is successfully used to solve a class of fractional Fredholm-Volterra integro-differential equations. The method is easy to implement, and applications are demonstrated through several illustrative examples. The numerical results are in excellent agreement with those obtained in Meng et al. (2015) and Rahimkhani et al. (2017). Note that we have computed the numerical results using the Maple package.

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