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Toeplitz Determinant for a Subclass of Tilted Starlike Functions with Respect to Conjugate Points

(Penentu Toeplitz untuk Subkelas Fungsi Bak Bintang Miring terhadap Titik-titik Konjugat)

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ABSTRACT

Let $S_c^*(\alpha,\delta,A,B)$ *denote the class of analytic and univalent functions in an open unit disk* $E=\{z\in\mathbb{C}:|z|<1\}$ *of the form* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *and satisfy* $\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}}\prec\frac{1+Az}{1+Bz}$ *where* $g(z)=\frac{f(z)+\overline{f(\bar{z})}}{2}$, $t_{\alpha\delta}=\cos\alpha-\delta>0$, $0\le\delta<1$, $|\alpha|<\frac{\pi}{2}$ *and* $-1\le B<A\le1$. *This paper presents the coefficient bounds for functions in* $S_c^*(\alpha,\delta,A,B)$ *using symmetric Toeplitz determinants* $T_2(2)$, $T_3(2)$ *and* $T_3(1)$. *The results obtained generalize the results for some existing subclasses in the literature.*

Keywords: Coefficient bounds; starlike functions with respect to conjugate points; Toeplitz determinant

ABSTRAK

Andaikan $S_c^*(\alpha,\delta,A,B)$ *sebagai kelas fungsi analisaan dan univalen dalam cakera unit terbuka* $E=\{z\in\mathbb{C}:|z|<1\}$ *dalam bentuk* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *dan memenuhi syarat* $\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}}\prec\frac{1+Az}{1+Bz}$ *dengan* $g(z)=\frac{f(z)+\overline{f(\bar{z})}}{2}$, $t_{\alpha\delta}=\cos\alpha-\delta>0$, $0\le\delta<1$, $|\alpha|<\frac{\pi}{2}$ *dan* $-1\le B<A\le1$. *Makalah ini membentangkan batas-batas pekali bagi fungsi dalam* $S_c^*(\alpha,\delta,A,B)$ *menggunakan penentu-penentu* Toeplitz $T_2(2)$, $T_3(2)$ dan $T_3(1)$. *Keputusan yang diperoleh mengitlak keputusan beberapa subkelas dalam kajian lepas.*

Kata kunci: Batas-batas pekali; fungsi bak bintang terhadap titik-titik konjugat; penentu Toeplitz

INTRODUCTION

Let H be the class of functions ω which are analytic in an open unit disk $E=\{z\in\mathbb{C}:|z|<1\}$ given by

$$\omega(z)=\sum_{k=2}^{\infty}b_kz^k \tag{1}$$

and satisfying the conditions $\omega(0)=0$, $|\omega(z)|<1$, $z\in E$. Let two functions $F(z)$ and $G(z)$ be analytic in *E*. If there exists a Schwarz function $\omega\in H$ which is analytic in E such that $F(z)=G(\omega(z))$, then $F(z)\prec G(z)$. The symbol "$\prec$" denotes the subordination. Further, if $G(z)$ is univalent in E, then $F(z)\prec G(z)\Leftrightarrow F(0)=G(0)$ and $F(E)=G(E)$.

Let A be the class of functions f which are analytic and univalent in E and of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ z\in E. \tag{2}$$

We denote by *S* the subclass of A consisting of all univalent functions in *E*. The well-known subclasses of *S* namely starlike, convex and close-to-convex functions, respectively denoted by S^*, *K* and *C*.

In 1987, El-Ashwah and Thomas defined the class of starlike functions with respect to conjugate points S_C^* consisting of functions of the form (2) and satisfying the condition

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\right\}>0,\ z\in E. \tag{3}$$

In 1991, Halim defined the class $S_C^*(\delta)$ consisting of functions of the form (2) and satisfying the condition

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\right\}>\delta,\ 0\le\delta<1,\ z\in E. \quad (4)$$

In 2009, Dahhar and Janteng introduced the class $S_C^*(A, B)$ consisting of functions of the form (2) and satisfying the condition

$$\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\prec\frac{1+Az}{1+Bz},\ -1\le B<A\le 1,\ z\in E. \quad (5)$$

By definition of subordination, from (5), it follows that $f\in S_C^*(A,B)$ if and only if

$$\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}=\frac{1+A\omega(z)}{1+B\omega(z)},\ \omega\in H. \quad (6)$$

Motivated from the work of El-Ashwah and Thomas (1987), Halim (1991) and Dahhar and Janteng (2009), Wahid et al. (2015) introduced the subclass of tilted starlike functions with respect to conjugate points of order δ $S_C^*(\alpha,\delta,A,B)$ consisting of functions given by (2) and satisfying the condition

$$\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}}\prec\frac{1+Az}{1+Bz}, \quad (7)$$

where $g(z)=\dfrac{f(z)+\overline{f(\bar{z})}}{2}$, $t_{\alpha\delta}=\cos\alpha-\delta>0,\ 0\le\delta<1$, $|\alpha|<\dfrac{\pi}{2}$ and $-1\le B<A\le 1$.

By definition of subordination, from (7), it follows that $f\in S_C^*(\alpha,\delta,A,B)$ if and only if

$$\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}}=\frac{1+A\omega(z)}{1+B\omega(z)},\ \omega\in H. \quad (8)$$

Obviously, the functions in these classes $S_C^*(\delta)$, $S_C^*(A,B)$ and $S_C^*(\alpha,\delta,A,B)$, respectively, are the subclass of the classes $S_C^*=S_C^*(0)$, $S_C^*=S_C^*(1,-1)$ and $S_C^*=S_C^*(0,0,1,-1)$.

It is known that the Toeplitz matrices are closely related to the Hankel matrices and one of the well-studied classes of structured matrices. The Toeplitz matrices have constant entries along the diagonals whereas the Hankel matrices have constant entries along the reverse diagonals. The Toeplitz matrices have many applications in the branches of pure and applied mathematics that led to some of the major developments of studies related to Toeplitz determinants, Toeplitz kernel, Toeplitz operators and q-deformed Toeplitz matrices (see Ye and Lim (2016) for details). Recall that the Hankel determinant $H_q(n),\ n,q\ge 1$ for f with the form as in (2) was defined by Pommerenke (1966) and Pommerenke (1967) as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}.$$

Thomas and Halim (2016) introduced the symmetric Toeplitz determinant $T_q(n),\ n,q\ge 1$ for f with the form as in (2) given by

$$T_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_n & \cdots & a_{n+q-2}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n\end{vmatrix},\ a_1=1.$$

In particular, we have

$$T_2(2)=\begin{vmatrix} a_2 & a_3\\ a_3 & a_2\end{vmatrix}=a_2^2-a_3^2,$$

$$T_3(1)=\begin{vmatrix} 1 & a_2 & a_3\\ a_2 & 1 & a_2\\ a_3 & a_2 & 1\end{vmatrix}=1\left(1-a_2^2\right)-a_2\left(a_2-a_2a_3\right)+a_3\left(a_2^2-a_3\right),$$

$$T_3(2)=\begin{vmatrix} a_2 & a_3 & a_4\\ a_3 & a_2 & a_3\\ a_4 & a_2 & a_2\end{vmatrix}=a_2\left(a_2^2-a_2a_3\right)-a_3\left(a_2a_3-a_3a_4\right)+$$

$$a_4\left(a_2a_3-a_2a_4\right),$$

and so on.

In recent years, the studies in a problem of estimating the coefficient bounds for the Toeplitz determinants for the class of univalent functions and its subclasses have been done by some researchers such as Al-Khafaji et al. (2020), Ali et al. (2018), Radhika et al. (2018, 2016), Ramachandran and Kavitha (2017), Sivasubramanian et al. (2016), Srivastava et al. (2019), Thomas and Halim (2016), and Zhang et al. (2019). However, there was no study of finding estimates for the Toeplitz determinants $T_2(2), T_3(2)$ and $T_3(1)$ for the subclasses introduced by El-Ashwah and Thomas (1987), Halim (1991) and Dahhar and Janteng (2009). In this paper, we determine the coefficient bounds for the Toeplitz determinants $T_2(2)$, $T_3(1)$ and $T_3(2)$ for the class $S_C^*(\alpha,\delta,A,B)$ consisting the functions given by (2). We also give some results for the subclasses introduced by El-Ashwah and Thomas (1987), Halim (1991) and Dahhar and Janteng (2009).

We shall state the following lemmas to prove our main results.

PRELIMINARY RESULTS

Let *P* be the class of functions *p* of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{9}$$

that is analytic in *E* and satisfying the condition $\operatorname{Re} p(z) > 0,\ z \in E$. Functions in *P* are sometimes called Carathéodory functions.

Lemma 1 (Duren 1983) (p. 41) For a function $p \in P$ of the form (9), the sharp inequality $|p_n| \le 2$ holds for each $n \ge 1$. Equality holds for the function $p(z) = \frac{1+z}{1-z}$.

Lemma 2 (Efraimidis 2016) Let $p \in P$ of the form (9) and $\mu \in \mathbb{C}$. Then

$$|p_n - \mu p_k p_{n-k}| \le 2\max\{1, |2\mu - 1|\},\ 1 \le k \le n-1.$$

If $|2\mu - 1| \ge 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$ or its rotations.

MAIN RESULTS

Theorem 1 If $f \in S_C^*(\alpha, \delta, A, B)$, then

$$|a_2^2 - a_3^2| \le \frac{T^2}{64}\Bigg\{64 + 16\sqrt{\left[2(1+B)T - \cos\alpha\left((1+B)^2 + T^2\right)\right]^2 + \left(-\sin\alpha\left((1+B)^2 - T^2\right)\right)^2}$$
$$+16\sqrt{(1+2B)^2 - 4(1+2B)T\cos\alpha + 4T^2}\Bigg\}$$

where $T = (A-B)t_{\alpha\delta}$ and $t_{\alpha\delta} = \cos\alpha - \delta$. The inequality is sharp.

Proof. As $f \in S_C^*(\alpha, \delta, A, B)$, so from (8)

$$\left\{e^{i\alpha}\frac{zf'(z)}{g(z)} - \delta - i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}} = \frac{1 + A\omega(z)}{1 + B\omega(z)},\ \omega \in H \tag{10}$$

where $g(z) = \frac{f(z) + \overline{f(\bar{z})}}{2}$ and $t_{\alpha\delta} = \cos\alpha - \delta$.

Now, let

$$h(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + \sum_{n=1}^{\infty} k_n z^n.$$

We have $h \in P$ and

$$\omega(z) = \frac{h(z) - 1}{h(z) + 1}. \tag{11}$$

Thus, by using (11), from (10), we have

$$e^{i\alpha}\frac{zf'(z)}{g(z)} = \frac{\left[e^{i\alpha}(1-B) - T\right] + h(z)\left[e^{i\alpha}(1+B) + T\right]}{1 - B + h(z)(1+B)} \tag{12}$$

where $T = (A-B)t_{\alpha\delta}$.

Using the series expansion in (12), we get

$$\begin{aligned} &e^{i\alpha}(1-B)\left(z + 2a_2z^2 + 3a_3z^3 + \cdots\right) \\ &+ e^{i\alpha}(1+B)\left(z + 2a_2z^2 + 3a_3z^3 + \cdots\right)\left(1 + k_1z + k_2z^2 + \cdots\right) \\ &= \left[e^{i\alpha}(1-B) - T\right]\left(z + a_2z^2 + a_3z^3 + \cdots\right) \\ &+ \left[e^{i\alpha}(1+B) + T\right]\left(z + a_2z^2 + a_3z^3 + \cdots\right)\left(1 + k_1z + k_2z^2 + \cdots\right). \end{aligned} \tag{13}$$

Equating the coefficients of z^2, z^3 and z^4 in the expansion of (13) give us

$$a_2 = \frac{k_1Te^{-i\alpha}}{2}, \tag{14}$$

$$a_3 = \frac{2k_2Te^{-i\alpha} + k_1^2T^2e^{-2i\alpha} - (1+B)k_1^2Te^{-i\alpha}}{8}, \tag{15}$$

and

$$\begin{aligned} a_4 = &\frac{8k_3Te^{-i\alpha} + 6k_1k_2T^2e^{-2i\alpha} - 8(1+B)k_1k_2Te^{-i\alpha} + k_1^3T^3e^{-3i\alpha}}{48} \\ &- \frac{3(1+B)k_1^3T^2e^{-2i\alpha} + 2(1+B)^2k_1^3Te^{-i\alpha}}{48}. \end{aligned} \tag{16}$$

Squaring (14) and (15), respectively, we get

$$a_2^2 = \frac{k_1^2T^2e^{-2i\alpha}}{4}, \tag{17}$$

$$\begin{aligned} a_3^2 = &\frac{4k_2^2T^2e^{-2i\alpha} + (1+B)^2k_1^4T^2e^{-2i\alpha} - 4(1+B)k_1^2k_2T^2e^{-2i\alpha}}{64} \\ &- \frac{2(1+B)k_1^4T^3e^{-3i\alpha} + 4k_1^2k_2T^3e^{-3i\alpha} + k_1^4T^4e^{-4i\alpha}}{64}, \end{aligned} \tag{18}$$

and so by some simple computations, the determinant $T_2(2)$ can be written as

$$\begin{aligned} |a_2^2 - a_3^2| = &\Bigg|\frac{16k_1^2T^2e^{-2i\alpha} - (1+B)^2k_1^4T^2e^{-2i\alpha} + 2(1+B)k_1^4T^3e^{-3i\alpha} - k_1^4T^4e^{-4i\alpha}}{64} \\ &- \frac{4k_2^2T^2e^{-2i\alpha} + 4(1+B)k_1^2k_2T^2e^{-2i\alpha} - 4k_1^2k_2T^3e^{-3i\alpha}}{64}\Bigg| \\ &= \frac{|T^2e^{-2i\alpha}|}{64}\Big|16k_1^2 + k_1^4\left(-(1+B)^2 + 2(1+B)Te^{-i\alpha} - T^2e^{-2i\alpha}\right) \\ &-4k_2^2 + 4k_1^2k_2\left(-Te^{-i\alpha} + (1+B)\right)\Big| \\ &= \frac{|T^2e^{-2i\alpha}|}{64}\Big|16k_1^2 + k_1^4\left(-(1+B)^2 + 2(1+B)Te^{-i\alpha} - T^2e^{-2i\alpha}\right) - \\ &4k_2\left[k_2 - \lambda k_1^2\right]\Big| \end{aligned} \tag{19}$$

where $\lambda = -Te^{-i\alpha} + (1+B)$.
Consequently, by the triangle inequality

$$\left|a_2^2 - a_3^2\right| \le \frac{T^2}{64}\Big(16\left|k_1^2\right| + \left|k_1^4\right|\left|-(1+B)^2 + 2(1+B)Te^{-i\alpha} - T^2 e^{-2i\alpha}\right| + 4\left|k_2\right|\left|k_2 - \lambda k_1^2\right|\Big) \tag{20}$$

which gives

$$\begin{aligned}
&\left|2(1+B)Te^{-i\alpha} - (1+B)^2 - T^2 e^{-2i\alpha}\right| \\
&= \left|e^{-i\alpha}\right|\left|2(1+B)T - (1+B)^2 e^{i\alpha} - T^2 e^{-i\alpha}\right| \\
&= \left|2(1+B)T - \cos\alpha\left((1+B)^2 + T^2\right) - i\sin\alpha\left((1+B)^2 - T^2\right)\right| \\
&= \sqrt{\left[2(1+B)T - \cos\alpha\left((1+B)^2 + T^2\right)\right]^2 + \left[-\sin\alpha\left((1+B)^2 - T^2\right)\right]^2}.
\end{aligned}$$

By Lemma 2,

$$\left|k_2 - \lambda k_1^2\right| \le 2\max\left\{1, \left|2\lambda - 1\right|\right\}$$

where

$$\begin{aligned}
\left|2\lambda - 1\right| &= \left|2\left(-Te^{-i\alpha} + (1+B)\right) - 1\right| \\
&= \left|1 + 2B - 2T\cos\alpha + 2Ti\sin\alpha\right| \\
&= \sqrt{(1 + 2B - 2T\cos\alpha)^2 + (2T\sin\alpha)^2} \\
&= \sqrt{(1+2B)^2 - 4(1+2B)T\cos\alpha + 4T^2}.
\end{aligned}$$

Thus,

$$\left|k_2 - \lambda k_1^2\right| \le 2\max\left\{1, \sqrt{(1+2B)^2 - 4(1+2B)T\cos\alpha + 4T^2}\right\}. \tag{21}$$

Applying Lemma 1 and using (21), from (20), we obtain

$$\begin{aligned}
\left|a_2^2 - a_3^2\right| \le \frac{T^2}{64}\Big\{&64 + 16\sqrt{\left[2(1+B)T - \cos\alpha\left((1+B)^2 + T^2\right)\right]^2 + \left[-\sin\alpha\left((1+B)^2 - T^2\right)\right]^2} \\
&+ 16\sqrt{(1+2B)^2 - 4(1+2B)T\cos\alpha + 4T^2}\Big\}.
\end{aligned}$$

The result is sharp for the function given by $\left\{e^{i\alpha}\frac{zf'(z)}{g(z)} - \delta - i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}} = \frac{1+z}{1-z}$. This completes the proof of Theorem 1.

Theorem 2 If $f \in S_C^*(\alpha, \delta, A, B)$, then

$$\begin{aligned}
\left|T_3(2)\right| \le \frac{T^3}{4608}\Bigg\{&48 + 16 + 2\sqrt{64(1+B)^2 - 96(1+B)T\cos\alpha + 36T^2} \\
&\bullet\left[\frac{2\sqrt{\left[-3BT + \cos\alpha\left(2\left(B^2 - 1\right) + T^2\right)\right]^2 + \left[\sin\alpha\left(2\left(B^2 - 1\right) - T^2\right)\right]^2}}{\sqrt{16(1+B)^2 - 24(1+B)T\cos\alpha + 9T^2}}\right]\Bigg\} \\
&\bullet\Big\{96 + 48 + 16\sqrt{4(B+2)^2 - 12(B+2)T\cos\alpha + 9T^2} \\
&+ 16\sqrt{\left[3(1+B)T - \cos\alpha\left((1+B)^2 + 2T^2\right)\right]^2 + \left[\sin\alpha\left((1+B)^2 - 2T^2\right)\right]^2}\Big\}
\end{aligned}$$

where $T = (A - B)t_{\alpha\delta}$ and $t_{\alpha\delta} = \cos\alpha - \delta$. The inequality is sharp.

Proof. Upon simplification, the determinant $T_3(2)$ can be written as

$$\begin{aligned}
\left|T_3(2)\right| &= \left|\left(a_2 - a_4\right)\left(a_2^2 - 2a_3^2 + a_2 a_4\right)\right| \\
&\le \left|a_2 - a_4\right|\left|a_2^2 - 2a_3^2 + a_2 a_4\right|.
\end{aligned}$$

Using (14) and (16) yield

$$\begin{aligned}
a_2 - a_4 = \frac{k_1 Te^{-i\alpha}}{2} - \Bigg[&\frac{8k_3 Te^{-i\alpha} + 6k_1 k_2 T^2 e^{-2i\alpha} - 8(1+B)k_1 k_2 Te^{-i\alpha} + k_1^3 T^3 e^{-3i\alpha}}{48} \\
&- \frac{3(1+B)k_1^3 T^2 e^{-2i\alpha} + 2(1+B)^2 k_1^3 Te^{-i\alpha}}{48}\Bigg],
\end{aligned}$$

and so

$$\begin{aligned}
\left|a_2 - a_4\right| &= \Bigg|\frac{24k_1 Te^{-i\alpha} + 8(1+B)k_1 k_2 Te^{-i\alpha} - 6k_1 k_2 T^2 e^{-2i\alpha} - 8k_3 Te^{-i\alpha}}{48} \\
&\quad - \frac{2(1+B)^2 k_1^3 Te^{-i\alpha} + 3(1+B)k_1^3 T^2 e^{-2i\alpha} - k_1^3 T^3 e^{-3i\alpha}}{48}\Bigg| \\
&= \frac{\left|Te^{-i\alpha}\right|}{48}\Big|24k_1 - 8k_3 + k_1\Big[k_2\left(8(1+B) - 6Te^{-i\alpha}\right) \\
&\quad - k_1^2\left(-3(1+B)Te^{-i\alpha} + 2(1+B)^2 + T^2 e^{-2i\alpha}\right)\Big]\Big| \qquad (22) \\
&= \frac{\left|Te^{-i\alpha}\right|}{48}\Bigg|24k_1 - 8k_3 + k_1\left(8(1+B) - 6Te^{-i\alpha}\right) \\
&\quad \bullet\left[k_2 - k_1^2\frac{\left(-3(1+B)Te^{-i\alpha} + 2(1+B)^2 + T^2 e^{-2i\alpha}\right)}{\left(8(1+B) - 6Te^{-i\alpha}\right)}\right]\Bigg| \\
&= \frac{\left|Te^{-i\alpha}\right|}{48}\left|24k_1 - 8k_3 + k_1\left(8(1+B) - 6Te^{-i\alpha}\right)\left[k_2 - \gamma k_1^2\right]\right|
\end{aligned}$$

where $\gamma = \dfrac{-3(1+B)Te^{-i\alpha} + 2(1+B)^2 + T^2 e^{-2i\alpha}}{8(1+B) - 6Te^{-i\alpha}}$.

Consequently, by the triangle inequality

$$\left|a_2 - a_4\right| \le \frac{\left|Te^{-i\alpha}\right|}{48}\Big(24\left|k_1\right| + 8\left|k_3\right| + \left|k_1\right|\left|8(1+B) - 6Te^{-i\alpha}\right|\left|k_2 - \gamma k_1^2\right|\Big). \tag{23}$$

By using the similar approach as in the proof of Theorem 1, from (23), we get

$$\begin{aligned}
\left|a_2 - a_4\right| \le \frac{T}{48}\Bigg\{&48 + 16 + 2\left(\sqrt{64(1+B)^2 - 96(1+B)T\cos\alpha + 36T^2}\right) \\
&\bullet\left[\frac{2\sqrt{\left[-3BT + \cos\alpha\left(2\left(B^2 - 1\right) + T^2\right)\right]^2 + \left[\sin\alpha\left(2\left(B^2 - 1\right) - T^2\right)\right]^2}}{\sqrt{16(1+B)^2 - 24(1+B)T\cos\alpha + 9T^2}}\right]\Bigg\}.
\end{aligned} \tag{24}$$

Equations (14) - (18) together yield

$$
\left|a_2^2 - 2a_3^2 + a_2a_4\right| = \left|\frac{k_1^2T^2e^{-2i\alpha}}{4} - 2\left[\frac{4k_2^2T^2e^{-2i\alpha} + 4k_1^2k_2T^3e^{-3i\alpha} - 4(1+B)k_1^2k_2T^2e^{-2i\alpha} + k_1^4T^4e^{-4i\alpha}}{64}\right.\right.
$$

$$
\left. - \frac{2(1+B)k_1^4T^3e^{-3i\alpha} + (1+B)^2k_1^4T^2e^{-2i\alpha}}{64}\right] + \frac{8k_1k_3T^2e^{-2i\alpha} + 6k_1^2k_2T^3e^{-3i\alpha}}{96}
$$

$$
\left. - \frac{8(1+B)k_1^2k_2T^2e^{-2i\alpha} + k_1^4T^4e^{-4i\alpha} - 3(1+B)k_1^4T^3e^{-3i\alpha} + 2(1+B)^2k_1^4T^2e^{-2i\alpha}}{96}\right|
$$

$$
= \left|\frac{24k_1^2T^2e^{-2i\alpha} - 2k_1^4T^4e^{-4i\alpha} - (1+B)^2k_1^4T^2e^{-2i\alpha} + 3(1+B)k_1^4T^3e^{-3i\alpha}}{96}\right.
$$

$$
\left. + \frac{4(1+B)k_1^2k_2T^2e^{-2i\alpha} - 6k_1^2k_2T^3e^{-3i\alpha} - 12k_2^2T^2e^{-2i\alpha} + 8k_1k_3T^2e^{-2i\alpha}}{96}\right|.
$$

Further, by arranging the terms, we have

$$
\left|a_2^2 - 2a_3^2 + a_2a_4\right| = \left|\frac{T^2e^{-2i\alpha}\left[24k_1^2 - 12k_2^2 + k_1^4\left(3(1+B)Te^{-i\alpha} - (1+B)^2 - 2T^2e^{-2i\alpha}\right)\right.}{96}\right.
$$

$$
\left. + \frac{8k_1\left[k_3 - k_2k_1\frac{\left(-4(1+B)\right) + 6Te^{-i\alpha}}{8}\right]\Big]}{96}\right| \tag{25}
$$

$$
= \left|\frac{T^2e^{-2i\alpha}\left[24k_1^2 - 12k_2^2 + k_1^4\left(3(1+B)Te^{-i\alpha} - (1+B)^2 - 2T^2e^{-2i\alpha}\right)\right.}{96} + \frac{8k_1\left[k_3 - \chi k_2k_1\right]\Big]}{96}\right|
$$

where $\chi = \frac{-4(1+B) + 6Te^{-i\alpha}}{8}$.

Applying the triangle inequality, (25) yields

$$
\left|a_2^2 - 2a_3^2 + a_2a_4\right| \le \frac{\left|T^2e^{-2i\alpha}\right|}{96}\Big(24\left|k_1^2\right| + 12\left|k_2^2\right| + \left|k_1^4\right|\left|3(1+B)Te^{-i\alpha} - 2T^2e^{-2i\alpha}\right| + 8\left|k_1\right|\left|k_3 - \chi k_2k_1\right|\Big). \tag{26}
$$

By using the similar approach as in the proof of Theorem 1, from (26), we get

$$
\left|a_2^2 - 2a_3^2 + a_2a_4\right| \le \frac{T^2}{96}\left\{96 + 48 + 16\sqrt{4(B+2)^2 - 12(B+2)T\cos\alpha + 9T^2}\right. \tag{27}
$$

$$
\left. + 16\sqrt{\left[3(1+B)T - \cos\alpha\left((1+B)^2 + 2T^2\right)\right]^2 + \left[\sin\alpha\left((1+B)^2 - 2T^2\right)\right]^2}\right\}.
$$

Hence, using (24) and (27), we obtain

$$
\left|T_3(2)\right| \le \frac{T^3}{4608}\left\{48 + 16 + 2\sqrt{64(1+B)^2 - 96(1+B)T\cos\alpha + 36T^2}\right.
$$

$$
\left. \bullet\left[\frac{2\sqrt{\left[-3BT + \cos\alpha\left(2\left(B^2-1\right) + T^2\right)\right]^2 + \left[\sin\alpha\left(2\left(B^2-1\right) - T^2\right)\right]^2}}{\sqrt{16(1+B)^2 - 24(1+B)T\cos\alpha + 9T^2}}\right]\right\}
$$

$$
\bullet\left\{96 + 48 + 16\sqrt{4(B+2)^2 - 12(B+2)T\cos\alpha + 9T^2}\right.
$$

$$
\left. + 16\sqrt{\left[3(1+B)T - \cos\alpha\left((1+B)^2 + 2T^2\right)\right]^2 + \left[\sin\alpha\left((1+B)^2 - 2T^2\right)\right]^2}\right\}.
$$

The result is sharp for the function given by $\left\{e^{i\alpha}\frac{zf'(z)}{g(z)} - \delta - i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}} = \frac{1+z}{1-z}$. This completes the proof of Theorem 2.

Theorem 3 If $f \in S_C^*(\alpha, \delta, A, B)$, then

$$
\left|T_3(1)\right| \le 1 + 2T^2 + \left(\frac{T^2\sqrt{B^2 - 2BT\cos\alpha + T^2}\sqrt{B^2 + 6BT\cos\alpha + 9T^2}}{4}\right)
$$

where $T = (A - B)t_{\alpha\delta}$ and $t_{\alpha\delta} = \cos\alpha - \delta$. The inequality is sharp.

Proof. Expanding the determinant $T_3(1)$, we get

$$
\left|T_3(1)\right| = \left|1 - 2a_2^2 + 2a_2^2a_3 - a_3^2\right|
$$

$$
\le 1 + 2\left|a_2^2\right| + \left|a_3\right|\left|a_3 - 2a_2^2\right|.
$$

By using Lemma 1, from (17) yields

$$
\left|a_2^2\right| = \left|\frac{k_1^2T^2e^{-2i\alpha}}{4}\right| \le T^2 \tag{28}
$$

and by arranging the terms in (15), we have

$$
\left|a_3\right| = \left|\frac{2k_2Te^{-i\alpha} + k_1^2T^2e^{-2i\alpha} - (1+B)k_1^2Te^{-i\alpha}}{8}\right|
$$

$$
= \left|\frac{2Te^{-i\alpha}\left[k_2 - \frac{k_1^2}{2}\left((1+B) - Te^{-i\alpha}\right)\right]}{8}\right|
$$

$$
= \left|\frac{Te^{-i\alpha}\left[k_2 - \mu k_1^2\right]}{4}\right|
$$

where $\mu = \frac{(1+B) - Te^{-i\alpha}}{2}$.

Following the same approach as in Theorem 1, thus

$$
\left|a_3\right| \le \frac{T\sqrt{B^2 - 2BT\cos\alpha + T^2}}{2}. \tag{29}
$$

Equations (15) and (17) together yield

$$
\left|a_3 - 2a_2^2\right| = \left|\frac{2k_2Te^{-i\alpha} + k_1^2T^2e^{-2i\alpha} - (1+B)k_1^2Te^{-i\alpha}}{8} - \frac{2k_1^2T^2e^{-2i\alpha}}{4}\right|
$$

$$
= \left|\frac{2Te^{-i\alpha}\left[k_2 - \frac{k_1^2}{2}\left((1+B) + 3Te^{-i\alpha}\right)\right]}{8}\right| \tag{30}
$$

$$
= \left|\frac{Te^{-i\alpha}\left[k_2 - \nu k_1^2\right]}{4}\right|
$$

where

$$\nu = \frac{(1+B)+3Te^{-i\alpha}}{2}.$$

Following the same approach as in Theorem 1, we find that

$$\left|a_3 - 2a_2^{\,2}\right| \le \frac{T\sqrt{B^2+6BT\cos\alpha+9T^2}}{2}. \qquad (31)$$

Hence, by using Lemma 1, (28), (29) and (31), we obtain

$$\left|T_3(1)\right| \le 1+2T^2+\left(\frac{T^2\sqrt{B^2-2BT\cos\alpha+T^2}\sqrt{B^2+6BT\cos\alpha+9T^2}}{4}\right).$$

The result is sharp for the function given by $\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}}=\frac{1+z}{1-z}$. This completes the proof of Theorem 3.

Choosing and substituting the specific values for the parameters into the results of Theorem 1, Theorem 2 and Theorem 3 we obtain the following corollaries.

Corollary 1

a) For $f \in S_C^{\,*}(0,0,1,-1)$, we get $\left|a_2^{\,2}-a_3^{\,2}\right| \le 13.$

b) For $f \in S_C^{\,*}(0,0,1,-1)$, we get $\left|T_3(2)\right| \le 84.$

c) For $f \in S_C^{\,*}(0,0,1,-1)$, we get $\left|T_3(1)\right| \le 24.$

Corollary 2

a) For $f \in S_C^{\,*}(0,\delta,1,-1)$, we get

$$\left|a_2^{\,2}-a_3^{\,2}\right| \le (1-\delta)^2\left[4+\sqrt{16(1-\delta)^4}+\sqrt{1+8(1-\delta)+16(1-\delta)^2}\right].$$

b) For $f \in S_C^{\,*}(0,\delta,1,-1)$, we get

$$\left|T_3(2)\right| \le \frac{8(1-\delta)^3}{4608}\left\{48+16+24(1-\delta)\left[\frac{2\left(3(1-\delta)+2(1-\delta)^2\right)}{3(1-\delta)}\right]\right\}$$
$$\bullet\left\{96+48+32\sqrt{1-6(1-\delta)+9(1-\delta)^2}+128(1-\delta)^2\right\}.$$

c) For $f \in S_C^{\,*}(0,\delta,1,-1)$, we get

$$\left|T_3(1)\right| \le 1+8(1-\delta)^2+(1-\delta)^2\sqrt{1+4(1-\delta)+4(1-\delta)^2}\sqrt{1-12(1-\delta)+36(1-\delta)^2}.$$

Corollary 3

a) For $f \in S_C^{\,*}(0,0,A,B)$, we get

$$\left|a_2^{\,2}-a_3^{\,2}\right| \le \frac{(A-B)^2}{4}\left\{4+\sqrt{\left[2(1+B)(A-B)-(1+B)^2-(A-B)^2\right]^2}\right.$$
$$\left.+\sqrt{(1+2B)^2-4(1+2B)(A-B)+4(A-B)^2}\right\}.$$

b) For $f \in S_C^{\,*}(0,0,A,B)$, we get

$$\left|T_3(2)\right| \le \frac{(A-B)^3}{4608}\left\{48+16+2\sqrt{64(1+B)^2-96(1+B)(A-B)+36(A-B)^2}\right.$$
$$\left.\bullet\left[\frac{2\sqrt{\left[-3B(A-B)+2(B^2-1)+(A-B)^2\right]^2}}{\sqrt{16(1+B)^2-24(1+B)(A-B)+9(A-B)^2}}\right]\right\}$$
$$\bullet\left\{96+48+16\sqrt{4(B+2)^2-12(B+2)(A-B)+9(A-B)^2}\right.$$
$$\left.+16\sqrt{\left[3(1+B)(A-B)-(1+B)^2-2(A-B)^2\right]^2}\right\}.$$

c) For $f \in S_C^{\,*}(0,0,A,B)$, we get

$$\left|T_3(1)\right| \le 1+2(A-B)^2+\frac{1}{4}\left[(A-B)^2\sqrt{B^2-2B(A-B)+(A-B)^2}\right.$$
$$\left.\bullet\sqrt{B^2+6B(A-B)+9(A-B)^2}\right].$$

Corollaries 1, 2 and 3 show the coefficient bounds for the Toeplitz determinants for the subclasses introduced by El-Ashwah and Thomas (1987), Halim (1991) and Dahhar and Janteng (2009), respectively. The inequalities in Corollary 1 coincide with the results of Ali et al. (2018) for the class of starlike functions S^*. Notice that the result of coefficient bounds for the Toeplitz determinants for the class of starlike functions S^* and the class of starlike functions with respect to conjugate points $S_C^{\,*}$ are shown to be equivalent.

Conclusion

In this paper, we have obtained the coefficient bounds for the Toeplitz determinants $T_2(2)$, $T_3(1)$ and $T_3(2)$ for the class $S_C^{\,*}(\alpha,\delta,A,B)$. It is shown that by considering specific values for the parameters α, δ, A and B the results obtained can be reduced to the results for some existing subclasses.

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