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Maximum and Minimum Degree Energy of Commuting Graph for Dihedral Groups

(Tenaga Darjah Maksimum dan Minimum bagi Graf Kalis Tukar Tertib bagi Kumpulan Dwihedron)

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ABSTRACT

If G is a finite group and $Z(G)$ is the centre of G, then the commuting graph for G, denoted by Γ_G, has $G\backslash Z(G)$ as its vertices set with two distinct vertices v_p and v_q are adjacent if $v_p v_q = v_q v_p$. The degree of the vertex v_p of Γ_G, denoted by d_{v_p}, is the number of vertices adjacent to v_p. The maximum (or minimum) degree matrix of Γ_G is a square matrix whose (p,q)-th entry is $max\{d_{v_p}, d_{v_q}\}$ (or $min\{d_{v_p}, d_{v_q}\}$) whenever v_p and v_q are adjacent, otherwise, it is zero. This study presents the maximum and minimum degree energies of Γ_G for dihedral groups of order $2n$, D_{2n} by using the absolute eigenvalues of the corresponding maximum degree matrices ($MaxD(\Gamma_G)$) and minimum degree matrices ($MinD(\Gamma_G)$). Here, the comparison of maximum and minimum degree energy of Γ_G for D_{2n} is discussed by considering odd and even n cases. The result shows that for each case, both energies are non-negative even integers and always equal.

Keywords: Commuting graph; degree of vertex; dihedral group; energy of a graph

ABSTRAK

Jika G adalah suatu kumpulan terhingga dan $Z(G)$ adalah pusat bagi G, maka graf kalis tukar tertib bagi G, ditatatandakan dengan Γ_G, mempunyai $G\backslash Z(G)$ sebagai set bucunya dengan dua bucu berbeza v_p dan v_q adalah bersebelahan jika $v_p v_q = v_q v_p$. Darjah bucu v_p dalam Γ_G, ditatatandakan dengan d_{v_p}, adalah bilangan bucu bersebelahan dengan v_p. Matriks darjah maksimum (atau minimum) bagi Γ_G ialah matriks segiempat sama yang mana unsur ke-(p,q) adalah maks$\{d_{v_p}, d_{v_q}\}$ (atau min$\{d_{v_p}, d_{v_q}\}$) apabila v_p dan v_q bersebelahan, jika tidak, ia adalah sifar. Kajian ini mengemukakan tenaga darjah maksimum dan minimum Γ_G bagi kumpulan dwihedron berperingkat $2n$, D_{2n} dengan menggunakan nilai eigen mutlak bagi matriks darjah maksimum ($MaxD(\Gamma_G)$) dan matriks darjah minimum ($MinD(\Gamma_G)$) yang sepadan. Di sini, perbandingan tenaga darjah maksimum dan minimum Γ_G bagi D_{2n} dibincangkan dengan mempertimbangkan kes n ganjil dan genap. Hasilnya menunjukkan bahawa bagi setiap kes, kedua-dua tenaga adalah integer genap bukan negatif dan sentiasa sama.

Kata kunci: Darjah bucu; graf kalis tukar tertib; kumpulan dwihedron; tenaga graf

INTRODUCTION

The non-abelian dihedral group of order $2n$, $n \geq 3$, is defined as $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ (Aschbacher 2000). The centre of D_{2n}, $Z(D_{2n})$ is either $\{e\}$, if n is odd or $\{e, a^{\frac{n}{2}}\}$, if n is even. The centralizer of the element a^i in the group D_{2n} is $C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}$ and for the element $a^i b$ is either $C_{D_{2n}}(a^i b) = \{e, a^i b\}$, if n is odd or $C_{D_{2n}}(a^i b) = \left\{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\right\}$, if n is even.

Suppose now that G is a finite group and $Z(G)$ is the centre of G, then the commuting graph for G, denoted by Γ_G, has $G\backslash Z(G)$ as its vertices set with two distinct vertices v_p and v_q are adjacent if $v_p v_q = v_q v_p$ (Brauer & Fowler 1955). This graph is also related to the result from Bundy (2006), Nawawi (2013), Nawawi and Rowley (2015), and Nawawi, Husain and Ariffin (2019), who worked on the symmetric group of degree n, while for the symplectic group, it can be seen in Kasim and Nawawi (2021, 2018). The discussion of graphs related to semigroups is also found in Gheisari and Ahmad (2012).

Furthermore, Γ_G can be associated with the adjacency matrix of Γ_G, which is an $n \times n$ matrix $A(\Gamma_G) = [a_{pq}]$ whose entries a_{pq} are equal to one if there is an edge between v_p and v_q, and zero otherwise. The characteristic polynomial $P_{A(\Gamma G)}(\lambda)$ of Γ_G is defined by det (λI_n - $A(\Gamma_G$)), where I_n is an n×n identity matrix. The roots of an equation $P_{A(\Gamma G)}(\lambda) = 0$ are called the eigenvalues of Γ_G and they are labelled as $\lambda_1, \lambda_2, \ldots, \lambda_n$. The spectrum of Γ_G is the list of eigenvalues, which will be denoted by $Spec(\Gamma_G) = \left\{\lambda_1^{(k_1)}, \lambda_2^{(k_2)}, \ldots, \lambda_m^{(k_m)}\right\}$ with $\lambda_1, \lambda_2, \ldots, \lambda_m$ together with their respective multiplicities $k_1, k_2, \ldots, k_m$, where $m \leq n$. The energy of a graph is the sum of the absolute eigenvalues of the corresponding matrix (Gutman 1978).

This field of study is indeed significant and has its own contribution to several other areas. For instance, chemical graph theory is a branch of mathematical chemistry that applies graph theory to the mathematical modeling of chemical compounds (Trinajstic 1992), which then includes discussion of graph energies (Gutman 1978) by considering a chemical molecule as a graph and estimating the π-electron energy. This graph energy concept is also a useful tool for predicting the boiling points, the heat of vaporization, and critical temperatures of alkanes (Hosamani et al. 2017). Additionally, the ordering index for chemical structure coding indicates the correlation with boiling points (Wang & Ma 2016). Furthermore, graphs also play remarkable roles in solving network problems (Loh, Salleh & Sarmin 2014; Razak & Expert 2021).

Furthermore, associating matrices with several types of graphs with group elements as a set of vertices has become a very popular area of research at present. Abdussakir et al. (2019) described the energy of subgroup graphs for dihedral group by using the corresponding adjacency matrix. On the other hand, Romdhini and Nawawi (2022) presented the formula of energy of non-commuting graphs for dihedral group by considering the eigenvalues of the characteristic polynomial of the degree sum matrix. Moreover, Romdhini, Nawawi and Chen (2022) explored the degree exponent sum energy of commuting graphs for the same type of group.

Here, we focus on representing the commuting graphs for dihedral groups as the maximum degree matrix, defined by Adiga and Smitha (2009), and the minimum degree matrix, defined by Adiga and Swamy (2010). Taking the summation of the absolute eigenvalues computed from the corresponding matrices leads us to derive the formula of maximum and minimum degree energies of commuting graphs for dihedral groups, denoted by $E_{maxD}(\Gamma_G)$ and $E_{minD}(\Gamma_G)$, respectively.

This paper is organized as follows. We set forth several existing results which are relevant to our study in the next section. Subsequently, we provide the general formula of maximum and minimum degree energies for different subsets of dihedral groups accompanied by two examples of computation. In the end, we summarize the findings of this study in the last section.

PRELIMINARIES

We define d_{v_p} as the degree of v_p which is the number of vertices adjacent to v_p. The maximum degree matrix (*MaxD*) and minimum degree matrix (*MinD*) of order $n \times n$ associated with elements of $G \backslash Z(G)$ are given by $MaxD(\Gamma_G) = [maxd_{pq}]$, and $MinD(\Gamma_G) = [mind_{pq}]$, respectively, whose (p,q)-th entry are as follows:

$$maxd_{pq} = \begin{cases} max\{d_{v_p}, d_{v_q}\}, & \text{if } v_p \text{ and } v_q \text{ are adjacent} \\ 0, & \text{otherwise,} \end{cases}$$

$$mind_{pq} = \begin{cases} min\{d_{v_p}, d_{v_q}\}, & \text{if } v_p \text{ and } v_q \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, if every vertex in a graph has the same degree r, then the graph is called r-regular graph.

In this section, we include some previous results which are beneficial for the next section. The two following results are important in computing the characteristic polynomial of the commuting graph Γ_G.

Lemma 2.1 (Ramane & Shinde 2017) If a,b,c and d are real numbers, and J_n is an $n \times n$ matrix whose all elements are equal to 1, then the determinant of the $(n_1 + n_2) \times (n_1 + n_2)$ matrix of the form

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

can be simplified in an expression as

$$(\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}\big((\lambda - (n_1 - 1)a)(\lambda - (n_2 - 1)b) - n_1 n_2 cd\big),$$

where $1 \leq n_1, n_2 \leq n$ and $n_1 + n_2 = n$.

Theorem 2.1 (Gantmacher 1959) If a square matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the partition into four blocks, where A is a square non-singular matrix, then

$$|M| = \begin{vmatrix} A & B \\ 0 & D - CA^{-1}B \end{vmatrix} = |A||D - CA^{-1}B|.$$

Additionally, a graph of order n with every vertex having degree n - 1 is called a complete graph K_n and the complement of K_n, is denoted by $\overline{K}_n$ The following lemma is the result of the spectrum of K_n, which is useful in computing the maximum and minimum energy of Γ_G.

Lemma 2.2 (Brouwer & Haemers 2011) If K_n is the complete graph on n vertices, then its adjacency matrix is $J_n - I_n$ and the spectrum of K_n is $\{(n-1)^{(1)}, (-1)^{(n-1)}\}$.

This present work focuses on the dihedral groups of order $2n$, D_{2n} – the group which consists of the reflection and rotation movements of a regular n-gon to its original position. Let $G_1 = \{a^i : 1 \leq i \leq n\} \backslash Z(D_{2n})$ be the set of rotation elements of D_{2n} which are not members of $Z(D_{2n})$ and $G_2 = \{a^i b : 1 \leq i \leq n\}$ be the set of reflection elements of D_{2n}. The following is the result of the degree of each vertex of the commuting graph Γ_G for $G = G_1 \cup G_2$.

Theorem 2.2 (Romdhini, Nawawi & Chen 2022) Let Γ_G be the commuting graph for G, where $G = G_1 \cup G_2$. Then

1. The degree of a^i on Γ_G is $d_{a^i} = \begin{cases} n-2, & \text{if } n \text{ is odd} \\ n-3, & \text{if } n \text{ is even,} \end{cases}$
2. The degree of $a^i b$ on Γ_G is $d_{a^i b} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even.} \end{cases}$

Consequently, the isomorphism of commuting graph with common type of graphs can be seen in the following result:

Theorem 2.3 (Romdhini, Nawawi & Chen 2022) Let Γ_G be the commuting graph for G.

1. If $G = G_1$, then $\Gamma_G \cong K_m$, where $m = |G_1|$.
2. If $G = G_2$, then $\Gamma_G \cong \begin{cases} \overline{K}_n, & \text{if } n \text{ is odd} \\ 1-\text{regular graph}, & \text{if } n \text{ is even.} \end{cases}$

MAIN RESULTS

This section will present several results on the maximum and minimum degree energy of the commuting graph for the dihedral group of order $2n$. We divide n into two cases, namely when n is odd and n is even. This is strictly for $n \geq 3$ since the dihedral group is abelian for $n = 1$ and $n = 2$.

Theorem 3.1 Let Γ_G be the commuting graph for G.

1. If $G = G_1$, then $E_{MaxD}(\Gamma_G) = E_{MinD}(\Gamma_G)$
$$= \begin{cases} 2(n-2)^2, & \text{if } n \text{ is odd} \\ 2(n-3)^2, & \text{if } n \text{ is even.} \end{cases}$$
2. If $G = G_2$, then $E_{MaxD}(\Gamma_G) = E_{MinD}(\Gamma_G)$
$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ n, & \text{if } n \text{ is even.} \end{cases}$$

Proof.

When n is odd. From Theorem 2.3 (1), $\Gamma_G \cong K_m$, for $G = G_1$ and $m = |G_1| = n - 1$, removing e in $Z(D_{2n})$. Clearly every vertex of Γ_G has degree n - 2. Then we can construct $(n-1) \times (n-1)$ maximum degree matrices of Γ_G, $MaxD(\Gamma_G) = [maxd_{pq}]$ and minimum degree matrices of Γ_G, $MinD(\Gamma_G) = [mind_{pq}]$ whose (p,q)-th entry is $maxd_{pq} = \max\{n-2, n-2\} = n-2$, and $mind_{pq} = \min\{n-2, n-2\} = n-2$ for adjacent v_p and v_q, and 0 for diagonal entries.

$$MaxD(\Gamma_G) = MinD(\Gamma_G) =$$

$$\begin{bmatrix} 0 & n-2 & \cdots & n-2 \\ n-2 & 0 & \cdots & n-2 \\ \vdots & \vdots & \ddots & \vdots \\ n-2 & n-2 & \cdots & 0 \end{bmatrix} = (n-2)\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}.$$

In other words, $MaxD(\Gamma_G$ and $MinD(\Gamma_G)$ are the product of n - 2 and the adjacency matrix of K_{n-1}. Based on Lemma 2.2, $Spec(K_{n-1})$ is given by $\{(n-2)^{(1)}, (-1)^{(n-2)}\}$. Since the adjacency energy of K_{n-1} is $|n-2| + (n-2)|-1|$ $=2(n-2)$, the maximum and minimum degree energy of Γ_G will be $(n-2) \cdot 2(n-1) = 2(n-2)^2$.

When n is even. From Theorem 2.3 (1), $\Gamma_G \cong K_m$, for $G = G_1$ and $m = |G_1| = n - 2$, removing all elements in $Z(D_{2n})$ which are e and $a^{\frac{n}{2}}$. Then every vertex of Γ_G has degree n - 3. Then we can construct $(n-2) \times (n-2)$ maximum degree matrices of Γ_G, $MaxD(\Gamma_G) = [maxd_{pq}]$ and minimum degree matrices of Γ_G, $MinD(\Gamma_G) = [mind_{pq}]$ whose (p,q)-th entry is $maxd_{pq} = max\{n-3, n-3\} = n-3$ and $mind_{pq} = min\{n-3, n-3\} = n-3$, for adjacent v_p and v_q, and 0 for diagonal entries.

$$MaxD(\Gamma_G) = MinD(\Gamma_G) =$$

$$\begin{bmatrix} 0 & n-3 & \cdots & n-3 \\ n-3 & 0 & \cdots & n-3 \\ \vdots & \vdots & \ddots & \vdots \\ n-3 & n-3 & \cdots & 0 \end{bmatrix} = (n-3)\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}.$$

Thus, $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ are the product of n - 3 and the adjacency matrix of K_{n-2}. Again by using Lemma 2.2, $Spec(K_{n-2})$ is $\{(n-3)^{(1)}, (-1)^{(n-3)}\}$. Since the adjacency energy of K_{n-2} is $|n-3| + (n-3)|-1| =$ $2(n-3)$, the maximum and minimum degree energy of Γ_G will be $(n-3) \cdot 2(n-3) = 2(n-3)^2$.

When n is odd. From Theorem 2.3 (2), $\Gamma_G \cong \overline{K}_n$, for $G = G_2$, which clearly shows that all of the vertices have degree zero. Then we can construct an $n \times n$ maximum degree matrix of Γ_G, $MaxD(\Gamma_G) = [maxd_{pq}]$ and minimum degree matrix of Γ_G, $MinD(\Gamma_G) = [mind_{pq}]$ whose (p,q)-th entry is $maxd_{pq} = \max\{0,0\}0$ and $mind_{pq} = \min\{0,0\} = 0$, for adjacent v_p and v_q, and diagonal entries as well.

$$MaxD(\Gamma_G) = MinD(\Gamma_G) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

In other words, $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ are zero matrices. Thus, $E_{MaxD}(\Gamma_G) = E_{MinD}(\Gamma_G) = 0$.

When n is even. By Theorem 2.3 (2), for $G = G_2$, Γ_G is a regular graph with degree one due to there is only an edge between the vertices a^ib and $a^{\frac{n}{2}+i}b$. Then we can construct n×n maximum and minimum degree matrices of Γ_G as follows:

$$MaxD(\Gamma_G) = MinD(\Gamma_G) = \left[\begin{array}{ccc|ccc} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \\ \hline 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{array}\right] = \begin{bmatrix} 0_{\frac{n}{2}} & I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix}.$$

Consider the characteristic polynomial of $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$,

$$P_{MaxD(\Gamma_G)}(\lambda) = P_{MinD(\Gamma_G)}(\lambda) = \begin{vmatrix} \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{vmatrix}.$$

Suppose that R_i is the i-th row and R_i' is the new i-th row obtained from a row operation of $P_{MaxD}(\Gamma_G)(\lambda)$ and $P_{MinD(\Gamma G)}(\lambda)$. Also, let that C_i is the i-th row and C_i' is the new i-th column obtained from a column operation of $P_{MaxD(\Gamma G)}(\lambda)$ and $P_{MinD(\Gamma G)}(\lambda)$. To begin, we replace $R_{\frac{n}{2}+i}$ by $R'_{\frac{n}{2}+i} = R_{\frac{n}{2}+i} - R_i$, for every $1 \le i \le \frac{n}{2}$. Then, we note that $P_{MaxD(\Gamma G)}(\lambda)$ and $P_{MinD(\Gamma G)}(\lambda)$ can be expressed as $\begin{vmatrix} \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ -(\lambda+1)_{\frac{n}{2}} & (\lambda+1)I_{\frac{n}{2}} \end{vmatrix}$. Consequently, we replace C_i by $C_i' = C_i + C_{\frac{n}{2}+i}$, for every $1 \le i \le \frac{n}{2}$. Then we observe that $P_{MaxD(\Gamma G)}(\lambda)$ and $P_{MinD(\Gamma G)}(\lambda)$ can be expressed as $\begin{vmatrix} (\lambda-1)I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & (\lambda+1)I_{\frac{n}{2}} \end{vmatrix}$.

By using Theorem 2.1, it is the form of $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B| = |A||D|$, since $C = 0$. It implies that

$$P_{MaxD(\Gamma_G)}(\lambda) = P_{MinD(\Gamma_G)}(\lambda) =$$
$$\left|(\lambda-1)I_{\frac{n}{2}}\right| \left|(\lambda+1)I_{\frac{n}{2}}\right| = (\lambda+1)^{\frac{n}{2}}(\lambda-1)^{\frac{n}{2}}.$$

Therefore,

$$E_{MaxD}(\Gamma_G) = E_{MinD}(\Gamma_G) = \frac{n}{2}|-1| + \frac{n}{2}|1| = n.$$

Theorem 3.2 Let Γ_G be the commuting graph for $G = G_1 \cup G_2 \subset D_{2n}$, then the characteristic polynomial of maximum and minimum degree matrices for Γ_G are as follows.

1. For n is odd,

$$P_{MaxD(\Gamma_G)}(\lambda) = P_{MinD(\Gamma_G)}(\lambda)$$
$$= (\lambda + n - 2)^{n-2}\lambda^n(\lambda - (n-2)^2).$$

2. For n is even

$$P_{MaxD(\Gamma_G)}(\lambda) = P_{MinD(\Gamma_G)}(\lambda)$$
$$= (\lambda + n - 3)^{n-3}(\lambda - (n-3)^2)(\lambda+1)^{\frac{n}{2}}(\lambda-1)^{\frac{n}{2}}.$$

Proof

When n is odd, from Theorem 2.2, the degree of $a^i \in G$, $d_{a^i} = n - 2$ and the degree of $a^ib \in G$, $d_{a^ib} = 0$, for all $1 \le i \le n$. Then, by using the fact that $Z(D_{2n}) = \{e\}$, we have $2n - 1$ vertices for Γ_G. The set of vertices consists of $n - 1$ vertices of a^i, for $1 \le i \le n - 1$, and n vertices of a^ib, for $1 \le i \le n$. Then the maximum and minimum degree matrices for Γ_G are both of dimension $(2n - 1) \times (2n - 1)$, denoted by $MaxD(\Gamma_G) = [maxd_{pq}]$ and $MinD(\Gamma_G) = [mind_{pq}]$ whose (p,q)-th entry are:

(i) $maxd_{pq} = mind_{pq} = n - 2$, for $p \neq q$, and $1 \le p,q \le n - 1$;

(ii) $maxd_{pq} = mind_{pq} = 0$ for $1 \le p \le n - 1$ and $n \le q \le 2n - 1$;

(iii) $maxd_{pq} = mind_{pq} = 0$, for $n \le p \le 2n - 1$ and $1 \le q \le n - 1$;

(iv) $maxd_{pq} = mind_{pq} = 0$, for $p \neq q$, $n \le p,q \le 2n - 1$;

(v) $maxd_{pq} = mind_{pq} = 0$, for $p = q$.

We can construct $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ as follows:

$$MaxD(\Gamma_G) = MinD(\Gamma_G) = \begin{bmatrix} (n-2)(J_{n-1} - I_{n-1}) & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

Then we obtain the characteristic polynomial of $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ from the following determinant

$$P_{MaxD(\Gamma_G)}(\lambda) = P_{MinD(\Gamma_G)}(\lambda) =$$

$$\begin{vmatrix} (\lambda + n - 2)I_{n-1} - (n-2)J_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & 0_n \end{vmatrix}.$$

By using Lemma 2.1, with $n_1 = n - 1$, $n_2 = n$ and $a = n - 2$, $b = 0$, $c = 0$, $d = 0$, we get the required result.

When n is even, using Theorem 2.2, we know that $d_{a^i} = n - 3$ and $d_{a^ib} = 1$, for all $1 \le i \le n$. Then, using the fact that $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$, we have $2n - 2$ vertices for Γ_G, where $G = G_1 \cup G_2$. The set of vertices consists of $n - 2$ vertices of a^i, with $i \neq n, \frac{n}{2}$, and n vertices of a^ib, for $1 \le i \le n$. Then the maximum and minimum degree matrices for Γ_G are both of dimension $(2n - 2) \times (2n - 2)$, denoted by $MaxD(\Gamma_G) = [maxd_{pq}]$ and $MinD(\Gamma_G) = [mind_{pq}]$ whose (p,q)-th entry are:

(i) $maxd_{pq} = mind_{pq} = n - 3$, for $p \neq q$, and $1 \le p,q \le n - 2$;

(ii) $maxd_{pq} = mind_{pq} = 0$, for $1 \le p \le n - 2$ and $n - 1 \le q \le 2n - 2$;

(iii) $maxd_{pq} = mind_{pq} = 0$, for $n - 1 \le p \le 2n - 2$ and $1 \le q \le n - 2$;

(iv) $maxd_{pq} = mind_{pq} = 0$, for $p = q$.

Since there exists only an edge between the vertices a^ib and $a^{\frac{n}{2}+i}b$ in Γ_G, for all $1 \le i \le n$, then the next entries are:

(v) $maxd_{pq} = mind_{pq} = 0$, for $n - 1 \le p,q \le n - 2 + \frac{n}{2}$;

(vi) $maxd_{pq} = mind_{pq} = 1$, for $q = \frac{n}{2} + p$, $n - 1 \le p \le n - 2 + \frac{n}{2}$ and $n - 1 + \frac{n}{2} \le q \le 2n - 2$;

(vii) $maxd_{pq} = mind_{pq} = 0$, for $q \neq \frac{n}{2} + p$, $n - 1 \le p \le n - 2 + \frac{n}{2}$ and $n - 1 + \frac{n}{2} \le q \le 2n - 2$;

(viii) $maxd_{pq} = mind_{pq} = 1$, for $p = \frac{n}{2} + q$, $n - 1 + \frac{n}{2} \le p \le n - 2$ and $n - 1 \le q \le n - 2 + \frac{n}{2}$;

(ix) $maxd_{pq} = mind_{pq} = 0$, for $p \neq \frac{n}{2} + q$, $n - 1 + \frac{n}{2} \le p \le n - 2$ and $n - 1 \le q \le n - 2 + \frac{n}{2}$;

(x) $maxd_{pq} = mind_{pq} = 0$, for $n - 1 + \frac{n}{2} \le p,q \le 2n - 2$.

We can construct $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ as the following:

$$MaxD(\Gamma_G) = MinD(\Gamma_G) =$$

$$\left[\begin{array}{ccccc|cccc|ccc}
0 & n-3 & \cdots & n-3 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
n-3 & 0 & \cdots & n-3 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
n-3 & n-3 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0
\end{array}\right]$$

$$= \begin{bmatrix} (n-3)(J_{n-2} - I_{n-2}) & 0_{(n-2)\times\frac{n}{2}} & 0_{(n-2)\times\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & 0_{\frac{n}{2}} & I_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix}.$$

Then the characteristic polynomial of $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ is

$$P_{MaxD(\Gamma_G)}(\lambda) = P_{MinD(\Gamma_G)}(\lambda) =$$

$$\begin{vmatrix} (\lambda + n - 3)I_{n-2} - (n-3)J_{n-2} & 0_{(n-2)\times\frac{n}{2}} & 0_{(n-2)\times\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & -I_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{vmatrix}.$$

By using Theorem 2.1 with $A = (\lambda + n - 3)\, I_{n-2} - (n - 3)\, J_{n-2}$, $B = 0_{(n-2)} \times n)$, $C = 0_{n\times(n-2)}$, $D = \begin{bmatrix} \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{bmatrix}$, and $|A| \neq 0$, we get the form of $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B| = |A||D|$. Now we consider $|A|$. By using Lemma 2.1, with $n_1 = \frac{n-2}{2}$, $n_2 = \frac{n-2}{2}$ and $a = n - 3$, $b = n - 3$, $c = n - 3$, $d = n - 3$, we obtain that $|A| = (\lambda + n - 3)^{n-3}(\lambda - (n - 3)^2)$. Meanwhile for $|D|$, by the same argument of Theorem 3.1 (2) for even n, then $|D| = (\lambda + 1)^{\frac{n}{2}}(\lambda - 1)^{\frac{n}{2}}$. Therefore,

$$P_{MaxD(\Gamma_G)}(\lambda) = P_{MinD(\Gamma_G)}(\lambda) = |(\lambda + n - 3)I_{n-2} -$$

$$(n-3)J_{n-2}| \begin{vmatrix} \lambda I_{\frac{n}{2}\times\frac{n}{2}} & -I_{\frac{n}{2}\times\frac{n}{2}} \\ -I_{\frac{n}{2}\times\frac{n}{2}} & \lambda I_{\frac{n}{2}\times\frac{n}{2}} \end{vmatrix}$$

Theorem 3.3 Let Γ_G be the commuting graph for G, where $G = G_1 \cup G_2$, then the maximum and minimum degree energy for Γ_G is

$$E_{MaxD}(\Gamma_G) = E_{MinD}(\Gamma_G) = \begin{cases} 2(n-2)^2, & \text{if } n \text{ is odd} \\ 2(n-3)^2 + n, & \text{if } n \text{ is even.} \end{cases}$$

Proof

By Theorem 3.2 (1) for the odd n, the characteristic polynomial of $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ has three eigenvalues, with the first eigenvalue is $\lambda_1 = -(n - 2)$ of multiplicity $(n - 2)$, the second eigenvalue is $\lambda_2 = 0$ of multiplicity (n), and the last eigenvalue is $\lambda_3 = (n - 2)^2$ of multiplicity (1). Hence, the maximum and minimum degree energy for Γ_G is

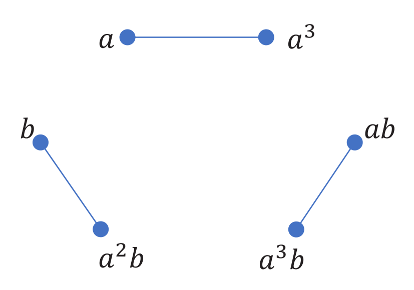
$$E_{MaxD}(\Gamma_G) = E_{MinD}(\Gamma_G) = (n-2)|-(n-2)| + (n)|0| + |(n-2)^2| = 2(n-2)^2.$$

By Theorem 3.2 (2) for the even n, the characteristic polynomial of $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ has four eigenvalues, with the first eigenvalue is $\lambda_1 = -(n - 3)$ of multiplicity $(n - 3)$, and the second eigenvalue $\lambda_1 = (n - 3)^2$ of multiplicity (1). The two last eigenvalues are $\lambda_3 = -1$ and $\lambda_3 = 1$ of multiplicity $\binom{n}{2}$ each. Thus, the maximum and minimum degree energy for Γ_G is

$$E_{MaxD}(\Gamma_G) = E_{MinD}(\Gamma_G) = (n-3)|-(n-3)| + |(n-3)^2| + \binom{n}{2}|-1| + \binom{n}{2}|1| = 2(n-3)^2 + n.$$

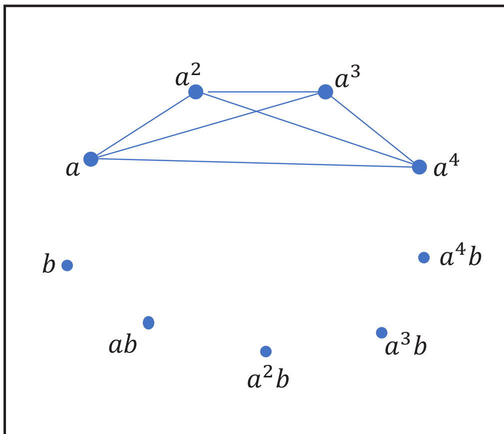
The maximum degree energy is equal to the minimum degree energy of commuting graph for dihedral groups of order 2n, D_{2n}, when $n = 4$ and $n = 5$ as illustrated in the following examples.

Example 1. Let Γ_G be the commuting graph for G, the maximum and minimum degree matrices of Γ_G as in Figure 1, where $G = G_1 \cup G_1 \subset D_8$, $G_1 = \{a, a^3\}$ and $G_2 = \{b, ab, a^2 b, a^3 b\}$.



$$MaxD(\Gamma_G) = MinD(\Gamma_G) = \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

FIGURE 1. [Commuting graph for $G = G_1 \cup G_2 \subset D_8$]



$$MaxD(\Gamma_G) = MinD(\Gamma_G)$$

$$= \left[\begin{array}{cccc|ccccc} 0 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

FIGURE 2. [Commuting graph for $G = G_1 \cup G_2 \subset D_{10}$]

By using Theorem 3.2 (2), the characteristic polynomial of $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ is $(\lambda + 1)^3 (\lambda - 1)^3$. It implies that the eigenvalues of $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ are $\lambda = -1$ with multiplicity (3) and $\lambda = 1$ with multiplicity (3). Therefore, $E_{MaxD}(\Gamma_G) = E_{MinD} (\Gamma_G) = 3 | - 1 | + 3 | 1 | = 6 = 2(4 - 3)^2 + 4$, conforming Theorem 3.3. for even n.

Example 2. Let Γ_G be the commuting graph for G, the maximum and minimum degree matrices of Γ_G as in Figure 2, where $G = G_1 \cup G_2 \subset D_{10}$, $G_1 = \{a, a^2, a^3, a^4\}$, and $G_2 = \{b, ab, a^2 b, a^3 b, a^4 b\}$.

By using Theorem 3.2.(1), the characteristic polynomial of $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ is $(\lambda + 3)^3 \lambda^5 (\lambda - 9)$. It implies that the eigenvalues of $MaxD(\Gamma_G)$ and $MinD(\Gamma_G)$ are $\lambda = -3$ with multiplicity (3), $\lambda = 0$ with multiplicity (5), and $\lambda = 9$ with multiplicity (1). Therefore, $E_{MaxD}(\Gamma_G) = E_{MinD}(\Gamma_G) = 3|-3|+5|0|+|9| = 18 = 2(5-2)^2$, conforming Theorem 3.3. for odd n.

Conclusions

We discuss the maximum and minimum degree energies of Γ_G for G being the set of non-center reflection elements, the set of rotation elements, and the union of the non-center reflection and rotation sets of D_{2n}, where $n \geq 3$. It is apparent that the maximum degree energy is always equal to the minimum degree energy of Γ_G, and they are always non-negative even integers.

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