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Comparison Analysis on the Coefficients of Variation of Two Independent Birnbaum-Saunders Distributions by Constructing Confidence Intervals for the Ratio of Coefficients of Variation

(Analisis Perbandingan Pekali Variasi Dua Taburan Birbaum-Saunders tak Bersandar dengan Membina Selang Keyakinan untuk Nisbah Pekali Variasi)

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ABSTRACT

The fatigue failure of materials can be investigated by applying the Birnbaum-Saunders (BS) distribution to fatigue failure datasets. The coefficient of variation (CV) is an important descriptive statistic that is widely used to measure the dispersion of data. In addition, for two independent datasets following BS distributions, the ratio of their CVs can be used to compare their CVs, especially when the difference is small, and constructing confidence intervals for this scenario is of interest in this study. Hence, we propose new confidence intervals for the ratio of the CVs from two BS distributions by using the bootstrap confidence interval (BCI), the fiducial generalized confidence interval (FGCI), a Bayesian credible interval (BayCI), and the highest posterior density (HPD) interval approaches. The performances of the proposed confidence intervals were compared with the generalized confidence interval (GCI) in terms of their coverage probabilities and average lengths via Monte Carlo simulations. The results indicate that the HPD interval outperformed the others when the coverage probabilities and the average lengths were both considered together. The efficacies of the proposed methods and GCI are illustrated using real datasets of the fatigue life of 6061-T6 aluminum coupons.

Keywords: Bayesian; Birnbaum-Saunders distribution; coefficients of variation; confidence interval; fatigue failure

ABSTRAK

Kegagalan lesu bahan boleh dikaji dengan menggunakan taburan Birnbaum-Saunders (BS) pada set data kegagalan lesu. Pekali variasi (CV) ialah statistik deskriptif penting yang digunakan secara meluas untuk mengukur serakan data. Di samping itu, untuk dua set data tak bersandar disebabkan taburan BS, nisbah CV mereka boleh digunakan untuk membandingkan CV mereka, terutamanya apabila perbezaannya kecil dan membina selang keyakinan untuk senario ini adalah penting dalam kajian ini. Oleh itu, kami mencadangkan selang keyakinan baharu untuk nisbah CV daripada dua taburan BS dengan menggunakan pendekatan selang keyakinan bootstrap (BCI), selang keyakinan umum fidusial (FGCI), selang boleh percaya Bayesian (BayCI) dan selang ketumpatan posterior tertinggi (HPD). Prestasi selang keyakinan yang dicadangkan telah dibandingkan dengan selang keyakinan umum (GCI) dari segi kebarangkalian liputan dan panjang purata melalui simulasi Monte Carlo. Keputusan menunjukkan bahawa selang HPD mengatasi yang lain apabila kebarangkalian liputan dan panjang purata kedua-duanya diambil kira secara bersama. Keberkesanan kaedah yang dicadangkan dan GCI diilustrasi menggunakan set data sebenar hayat lesu kupon aluminium 6061-T6.

Kata kunci: Bayesian; kegagalan lesu; pekali variasi; selang keyakinan; taburan Birnbaum-Saunders

INTRODUCTION

In engineering, the initiation and propagation of cracks in materials are caused by repeated stress cycles; the cracks can then grow to a critical size that eventually results in fatigue failure, which is one of the main reasons for mechanical failure. Therefore, satisfactory prior knowledge of the fatigue life of materials is important for keeping crack formation within acceptable limits, to predict the effects of changes in operational conditions, and to identify the cause of fatigue failure and instigate efficient mitigating action. In practice, positively skewed unimodal statistical distributions such as gamma, Weibull, Birnbaum-Saunders (BS), and lognormal are wildly applied for analyzing the fatigue life of materials, with the BS distribution being the most suitable because it was originally derived from features of the fatigue process (Marshall & Olkin 2007). The BS distribution has two positive parameters: α (the shape parameter) and β (the scale parameter) denoted as $BS(\alpha, \beta)$ developed under the assumption that the ultimate failure of an item occurs due to the development and growth of cracks in the material under cyclic loading. The BS distribution has attractive properties and a close relationship with the normal distribution (Birnbaum & Saunders 1969a). One of these is that a random variable following a BS distribution can be generated by transforming it to a standard normal distribution (Johnson et al. 1994). The BS distribution has been applied in many areas. For example, Birnbaum and Saunders (1996b) originally applied it to investigate the fatigue life of 6061-T6 aluminum coupons. Leiva et al. (2011) used the BS distribution with an unknown shift parameter to model wind energy flux. Recently, the BS distribution has been used for evaluating the effect of nanoparticles at different loading levels on the hardness of a commercially available polymeric bone cement.

Statistical inference with the parameters of the BS distribution has been published in many articles. Birnbaum and Saunders (1996b) originally investigated the maximum likelihood estimators (MLEs) of α and β. Subsequently, their asymptotic distributions were derived by Engelhardt et al. (1981). Ng et al. (2003) proposed modified moment estimators (MMEs) of α and β, and applied a bias-reduction method to mitigate the bias inherent in maximum likelihood and modified moment estimation. Sun (2009) formulated a confidence interval for scale parameter β. Wang (2012) considered generalized confidence intervals (GCI) for the shape parameter α, mean, quantiles, and reliability function of a BS distribution. Li and Xu (2016) presented fiducial inference for the parameters of BS distribution. Recently, Wang et al. (2016) proposed Bayesian estimators and confidence intervals for the parameters of a BS distribution using an efficient sampling algorithm via the generalized ratio-of-uniforms method.

One of the most useful descriptive statistical measures for describing the dispersion of data is the coefficient of variation (CV). It is defined as the standard deviation (σ) divided by the mean (μ): $\eta = \sigma/\mu$. For describing variation within data, the CV is more meaningful than the standard deviation because one can compare data from different distributions and/or units. For statistical inference using the CV for various distributions, please see Mahmoudvand and Hassani (2009), Niwitpong (2013), and Thangjai et al. (2021) works. When there are two independent populations, researchers may need to compare their coefficients of variation (CVs). Therefore, the problem of comparing two CVs is of interest. Several researchers have considered confidence intervals for the ratio of CVs for comparing two independent population CVs. For example, confidence intervals for the ratio of CVs of delta-lognormal distribution were proposed by Buntao and Niwitpong (2013) based on the generalized variable approach and the method of variance estimates recovery (MOVER). Subsequently, Sangnawakij et al. (2015) constructed confidence intervals for the ratio of CVs of gamma distributions using MOVER based on the Score and Wald intervals. Niwitpong and Wongkhao (2016) applied the GCI and MOVER to construct confidence intervals for the ratio of CVs of normal distributions with a known ratio of variances. Hasan and Krishnamoorthy (2017) developed confidence intervals for the ratio of CVs of lognormal distributions using the MOVER and fiducial approaches. Recently, Nam and Kwon (2017) improved confidence intervals for the ratio of CVs of lognormal distributions by using the Wald-type, Fieller-type, log, and MOVER methods. However, there were a few proposed inference procedures for the ratio of CVs of BS distributions. For example, Puggard et al. (2020) proposed the GCI approach for the ratio of CVs of BS distributions and compared its performance with bias-corrected percentile bootstrap (BCPB) and the bias-corrected and accelerated (BCa) confidence intervals; they recommended GCI since it produced coverage probabilities higher than or close to the nominal confidence level (those of BCPB and BCa were lower than it) and the shortest average lengths for all of the test scenarios. Therefore, the goal of the present study is to propose new confidence intervals for the ratio of CVs of two BS distributions using the bootstrap confidence interval (BCI) based on the constant-bias-correcting (CBC) parametric bootstrap method, the fiducial generalized confidence interval (FGCI), a Bayesian credible interval (BayCI) based on an efficient sampling algorithm via the generalized ratio-of-uniforms method, and the highest posterior density (HPD) interval. We then compared their performances with GCI, as recommended in the previous study of Puggard et al. (2020).

The rest of this study is organized as follows. The next section contains a summary of some of the properties of the BS distribution followed by an introduction to the methods for constructing confidence intervals for the ratio of CVs of BS distributions. Subsequent section presents the simulation study and results. This is followed by details of applying the proposed methods to datasets of fatigue lifetime of 6061-T6 aluminum coupons represented by Birnbaum and Saunders (1996b). Finally, conclusions are drawn in the last section.

METHODS

In this section, we review some of the background about the BS distribution and define GCI and the proposed methods: BCI, FCGI, BayCI, and the HPD interval for constructing the confidence interval for the ratio of the CVs from two BS distributions.

Suppose $X_{ij} = (X_{i1}, X_{i2}, ..., X_{in_i})$ $i = 1,2,\ j = 1,2,...,n_i$ comprise a vector of random samples from BS distributions with shape parameters α_i and scale parameters β_i; i.e., $X_{ij} \sim BS(\alpha_i, \beta_i)$, and let $\boldsymbol{x}_{ij} = (x_{i1}, x_{i2}, ..., x_{in_i})$ be the observed values of $\boldsymbol{X}_{ij}$. The corresponding cumulative distribution function (CDF) and probability density function are given by

$$F(x_{ij}) = \Phi\left[\frac{1}{\alpha_i}\left(\sqrt{\frac{x_{ij}}{\beta_i}} - \sqrt{\frac{\beta_i}{x_{ij}}}\right)\right], \tag{1}$$

and

$$f(x_{ij},\alpha_i,\beta_i) = \frac{1}{2\alpha_i\beta_i\sqrt{2\pi}}\left\{\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}} + \left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right\} exp \left[-\frac{1}{2\alpha_i^2}\left(\frac{x_{ij}}{\beta_i} + \frac{\beta_i}{x_{ij}} - 2\right)\right], \tag{2}$$

respectively, where $x_{ij} > 0$, $\alpha_i > 0$, $\beta_i > 0$, and $\Phi(.)$ is the standard normal CDF.

Note that if $X_{ij} \sim BS(\alpha_i, \beta_i)$, then

$$Y_{ij} = \frac{1}{2}\left(\sqrt{\frac{X_{ij}}{\beta_i}} - \sqrt{\frac{\beta_i}{X_{ij}}}\right)\ N(0, \frac{\alpha_i^2}{4}), \tag{3}$$

where $N(\mu, \sigma^2)$ refers to a normal distribution with mean μ and variance σ^2. Thus,

$$X_{ij} = \beta_i\left(1 + 2Y_{ij}^2 + 2Y_{ij}\sqrt{1+Y_{ij}^2}\right), \tag{4}$$

follows a BS distribution with parameters (α_i, β_i). Therefore, the above transformation can be used to generate a sample from a BS distribution. Using this transformation, the expected value and variance can be respectively obtained as

$$E(X_{ij}) = \beta_i\left(1 + \frac{1}{2}\alpha_i^2\right), \tag{5}$$

and

$$Var(X_{ij}) = (\alpha_i\beta_i)^2\left(1 + \frac{5}{4}\alpha_i^2\right). \tag{6}$$

In addition, if $X_{ij} \sim BS(\alpha_i, \beta_i)$, then $X_{ij}^{-1} \sim BS(\alpha_i, \beta_i^{-1})$ (Birnbaum & Saunders 1969a). Subsequently, the expected value and variance of X_{ij}^{-1} can be respectively expressed as

$$E(X_{ij}^{-1}) = \beta_i^{-1}\left(1 + \frac{1}{2}\alpha_i^2\right), \tag{7}$$

and

$$Var(X_{ij}^{-1}) = \alpha_i^2\beta_i^{-2}\left(1 + \frac{5}{4}\alpha_i^2\right). \tag{8}$$

By equations (5) and (6), the CVs denoted by η_i, $i = 1,2$ are obtained as

$$\eta_i = \frac{\sqrt{Var(X_{ij})}}{E(X_{ij})} = \frac{\alpha_i\sqrt{1+\frac{5}{4}\alpha_i^2}}{1+\frac{1}{2}\alpha_i^2}. \tag{9}$$

Therefore, the ratio of CVs denoted by ω becomes

$$\omega = \frac{\eta_1}{\eta_2} = \frac{\alpha_1\sqrt{1+\frac{5}{4}\alpha_1^2} \Big/ 1+\frac{1}{2}\alpha_1^2}{\alpha_2\sqrt{1+\frac{5}{4}\alpha_2^2} \Big/ 1+\frac{1}{2}\alpha_2^2}. \tag{10}$$

The following methods are applied to construct confidence intervals for ω.

GENERALIZED CONFIDENCE INTERVALS (GCI)

When the conventional pivotal quantity is either non-existent or difficult to obtain, one can use the generalized pivotal quantity (GPQ) to construct a confidence interval (Weerahandi 1993). For the BS distribution, the GPQs for β_i and α_i were proposed by Sun (2009) and Wang (2012), respectively. The GPQ for β_i is obtained as

$$R_{\beta_i} = \begin{cases} max(\beta_{i1}, \beta_{i2}), & if\ \ T_i \le 0; \\ min(\beta_{i1}, \beta_{i2}), & if\ \ T_i > 0, \end{cases}. \tag{11}$$

where i = 1,2 and $T_i \sim t(n_i - 1)$. Meanwhile, β_{i1} and β_{i2} are the solutions of the following quadratic equation:

$$\left[(n_i-1)J_i^2-\frac{1}{n_i}L_iT_i^2\right]\beta_i^2-2\left[(n_i-1)I_iJ_i-(1-I_iJ_i)T_i^2\right].$$
$$\beta_i+(n_i-1)I_i^2-\frac{1}{n_i}K_iT_i^2=0\,.$$

where $I_i=n_i^{-1}\sum_{j=1}^{n_i}\sqrt{X_{ij}}$, $J_i=n_i^{-1}\sum_{j=1}^{n_i}1\big/\sqrt{X_{ij}}$, $K_i=\sum_{j=1}^{n_i}\left(\sqrt{X_{ij}}-I_i\right)^2$ and $L_i=\sum_{j=1}^{n_i}\left(1\big/\sqrt{X_{ij}}-J_i\right)^2$. According to Wang (2012), the GPQ for α_i is

$$R_{\alpha_i}=\left[\frac{S_{i2}R_{\beta_i}^2-2n_iR_{\beta_i}+S_{i1}}{R_{\beta_i}\upsilon_i}\right]^{1/2}, \qquad (12)$$

where $S_{i1}=\sum_{j=1}^{n_i}X_{ij}$, $S_{i2}=\sum_{j=1}^{n_i}1\big/X_{ij}$, $\upsilon_i\sim\chi^2(n_i)$ and R_{β_i} is a GPQ of β_i. Subsequently, the GPQ for ω can be defined as

$$R_\omega=\frac{R_{\alpha_1}\sqrt{1+\frac{5}{4}R_{\alpha_1}}\Big/\left(1+\frac{1}{2}R_{\alpha_1}\right)}{R_{\alpha_2}\sqrt{1+\frac{5}{4}R_{\alpha_2}}\Big/\left(1+\frac{1}{2}R_{\alpha_2}\right)}. \qquad (13)$$

Based on GCI, the 100(1-γ)% confidence interval for ω is

$$CI_{GCI}=\left[L_\omega,U_\omega\right]=\left[R_\omega(\gamma/2),R_\omega(1-\gamma/2)\right]. \qquad (14)$$

where $R_\omega(v)$ is the 100% percentile of R_ω. *GCI* can be obtained by using the following algorithm.

The algorithm for GCI
1. For given $I_i, J_i, K_i, L_i, s_{i1}$ and s_{i2}, i = 1,2. 2. For k = 1 to K. 3. Generate T_i from a t-distribution with n_i - 1 degrees of freedom. 4. Compute R_{β_i} from (11) (if R_{β_i} < 0, regenerate $T_i \sim t(n_i - 1)$). 5. Generate υ_i from a Chi-squared distribution with n_i degrees of freedom. 6. Compute R_{α_i} from (12). 7. Compute R_ω from (13). 8. (End K loops). 9. Compute $R_\omega(\gamma/2)$ and $R_\omega(1 - \gamma/2)$.

BOOTSTRAP CONFIDENCE INTERVAL (BCI)

The bootstrap method introduced by Efron (1979) is a re-sampling technique based on the random selection of new samples from the original sample to construct a sampling distribution for a particular statistic. It has been used to construct confidence intervals for the parameter of interest (Chou et al. 2006; Kashif et al. 2017; Moslim et al. 2019). From (10), ω is a function of α_i, i = 1,2. Since the maximum likelihood estimators (MLEs) of α_i and β_i are biased (Ng et al. 2003), Lemonte et al. (2008) recommended that the best-performing method for reducing their bias is the CBC parametric bootstrap. Therefore, we applied it to estimate confidence interval for ω as follows.

Let $\boldsymbol{x}_{ij} = (x_{i1}, x_{i2}, ..., x_{in_i})$, i = 1,2, j = 1, 2, ..., n_i be an original random sample of size n_i drawn from $BS(\alpha_i, \beta_i)$ with distribution function $F = F_{\alpha_i,\beta_i}(x_{ij})$. The log-likelihood function of the BS distribution without the additive constant is given by

$$l(\alpha_i,\beta_i)=-n_ilog(\alpha_i\beta_i)+\sum_{j=1}^{n_i}log\left[\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}}+\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right]-\frac{1}{2\alpha_i^2}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{\beta_i}+\frac{\beta_i}{x_{ij}}-2\right). \qquad (15)$$

The Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-Newton nonlinear optimization algorithm can be used to compute the MLEs of α_i and β_i (denoted by $\hat{\alpha}_i$ and $\hat{\beta}_i$) by maximizing the log-likelihood function.

A bootstrap sample, $\boldsymbol{x}_{ij}^* = (x_{i1}^*, x_{i2}^*, ..., x_{in_i}^*)$, is a sample of size n_i obtained from $BS(\hat{\alpha}_i, \hat{\beta}_i)$ with distribution function $F = F_{\hat{\alpha}_i,\hat{\beta}_i}$. The estimator $\hat{\alpha}_i$ for the bootstrap sample, $\hat{\alpha}_i^*$ can be calculated by using the BFGS quasi-Newton nonlinear optimization algorithm. Suppose that B bootstrap samples are available, then B values of $\hat{\alpha}_i^*$ can be obtained and can be arranged in ascending order; i.e. $\hat{\alpha}_{i,(1)}^*, \hat{\alpha}_{i,(2)}^*, ..., \hat{\alpha}_{i,(B)}^*$, i = 1,2. The bias of estimator $\hat{\alpha}_i$ is given by

$$b_F(\hat{\alpha}_i,\alpha_i)=E_F\left[\hat{\alpha}_i-\alpha_i\right]=E_F(\hat{\alpha}_i)-\alpha_i, \qquad (16)$$

where the subscript F indicates that the expectation is taken with respect to F. The bootstrap estimator of the bias is obtained by replacing true distribution F with $F_{\hat{\alpha}_i,\hat{\beta}_i}$. Hence, the estimator of the bias can be defined as

$$b_{F_{\hat{\alpha}_i,\hat{\beta}_i}}(\hat{\alpha}_i,\alpha_i)=E_{F_{\hat{\alpha}_i,\hat{\beta}_i}}(\hat{\alpha}_i)-\alpha_i\,. \qquad (17)$$

Following this, $E_{F_{\hat{\alpha}_i,\hat{\beta}_i}}(\hat{\alpha}_i)$ is approximated by

$$\hat{\alpha}_i^{*(\cdot)}=1/B\sum_{k=1}^{B}\hat{\alpha}_{i,(k)}^*,\quad k=1,2,...,B\,. \qquad (18)$$

Based on B replications, the bootstrap bias estimate of $\hat{\alpha}_i$ is calculated as

$$\hat{b}_{F_{\hat{\alpha}_i,\hat{\beta}_i}}(\hat{\alpha}_i,\alpha_i)=\hat{\alpha}_i^{*(\cdot)}-\hat{\alpha}_i \,. \tag{19}$$

According to MacKinnon and Smith (1998), the corrected estimate for bootstrap sample (denoted as $\tilde{\alpha}_i, i=1,2$) can be obtained as

$$\tilde{\alpha}_i=\hat{\alpha}_i^{*}-2\hat{b}_{F_{\hat{\alpha}_i,\hat{\beta}_i}}(\hat{\alpha}_i,\alpha_i). \tag{20}$$

By Equation (10), the bootstrap estimator of ω becomes

$$\hat{\omega}=\frac{\tilde{\alpha}_1\sqrt{1+\frac{5}{4}\tilde{\alpha}_1^2}\Big/\left(1+\frac{1}{2}\tilde{\alpha}_1^2\right)}{\tilde{\alpha}_2\sqrt{1+\frac{5}{4}\tilde{\alpha}_2^2}\Big/\left(1+\frac{1}{2}\tilde{\alpha}_2^2\right)}. \tag{21}$$

Thus, based on BCI, the 100(1-γ)% confidence interval for ω is

$$CI_{BCI}=\left[L_\omega,U_\omega\right]=\left[\hat{\omega}(\gamma/2),\hat{\omega}(1-\gamma/2)\right], \tag{22}$$

where $\hat{\omega}(\nu)$ is the 100% percentile of $\hat{\omega}$. BCI can be obtained by using the following algorithm.

The algorithm for BCI

1. Compute MLE estimator $\hat{\alpha}_i$ and $\hat{\beta}_i$ from the original sample by applying BFGS. 2. For $k=1$ to B. 3. Generate $\boldsymbol{x}_{ij}^{*}$ from the BS distribution with parameter $\hat{\alpha}_i$ and $\hat{\beta}_i$. 4. Compute the bootstrap MLE estimator $\hat{\alpha}_i^{*}$ by applying BFGS. 5. (End B loops). 6. Compute $\hat{\alpha}_i^{*(\cdot)}$ and $\hat{b}_{F_{\hat{\alpha}_i,\hat{\beta}_i}}(\hat{\alpha}_i,\alpha_i)$ from (18) and (19), respectively. 7. Compute $\tilde{\alpha}_i$ from (20). 8. Compute $\hat{\omega}$ from (21). 9. Compute $\hat{\omega}$ (γ/2) and $\hat{\omega}$ (1 - γ/2).

FIDUCIAL GENERALIZED CONFIDENCE INTERVAL (FGCI)

The original idea of generalized fiducial inference can be traced back to Hannig (2009). Suppose that data Z and model parameter $\vartheta\in\Xi$ have the following functional relationship:

$$Z=H(\vartheta,U) \tag{23}$$

where $H(\cdot,\cdot)$ is the structural equation and U is a random variable with a known distribution that is free of parameters. In general, the inverse of the structural (23) does not exist, and so the solution for this problem proposed by Hannig (2013, 2009) is considered. After denoting H = (H_1, H_2, ..., H_n) as the structural equation, $Z_i=H_i(\vartheta,U)$, for i = 1,2, ..., n. Suppose that $\boldsymbol{U}=(U_1,U_2,...,U_n)$ are independent identically distributed (i.i.d.) samples from a uniform (0,1) distribution and that parameter $\vartheta\in\Xi\subseteq P$ is p-dimensional. Therefore, the generalized fiducial distribution proposed by Hannig (2013) is completely continuous with density

$$r(\vartheta)=\frac{L(z,\vartheta)J(z,\vartheta)}{\int_{\Xi}L(z,\vartheta')J(z,\vartheta')d\vartheta'}, \tag{24}$$

where $L(z,\vartheta)$ is the likelihood function, and

$$J(z,\vartheta)=\sum_{\substack{i=(i_1,\dots,i_p)\\ 1\le i_1<\dots<i_p\le n}}\left|\det\left(\left(\frac{d}{dz}\boldsymbol{H}^{-1}(z,\vartheta)\right)^{-1}\frac{d}{d\vartheta}\boldsymbol{H}^{-1}(z,\vartheta)\right)_i\right|, \tag{25}$$

where the sum is taken over all subsets of indexes $i=(1\le i_1<\dots<i_p\le n)\subset\{1,\dots,n\}$, $d\boldsymbol{H}^{-1}(z,\vartheta)/dz$ and $d\boldsymbol{H}^{-1}(z,\vartheta)/d\vartheta$ are $n\times n$ and $n\times p$ Jacobian matrices, respectively. Thus, we can obtain $n\times p$ matrix K and $p\times p$ matrix $(K)_i$ containing rows i_1, ..., i_p of K. Moreover, if z from a completely continuous distribution is i.i.d. with cumulative distribution function $F_\vartheta(z)$, then $\boldsymbol{H}^{-1}=(F_\vartheta(Z_1),\dots,F_\vartheta(Z_n))$ (Hannig 2009).

Let $\boldsymbol{X}_{ij}=(X_{i1},X_{i2},\dots,X_{in_i}), i=1,2, j=1,2,\dots,n_i$ follow $BS(\alpha_i,\beta_i)$, then the likelihood function is given by

$$L(\boldsymbol{x}_{ij}\mid\alpha_i,\beta_i)\propto\frac{1}{\alpha_i^{n_i}\beta_i^{n_i}}\prod_{j=1}^{n_i}\left[\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}}+\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right]exp\left[-\sum_{j=1}^{n_i}\frac{1}{2\alpha_i^2}\left(\frac{x_{ij}}{\beta_i}+\frac{\beta_i}{x_{ij}}-2\right)\right]. \tag{26}$$

By using (25), Li and Xu (2016) showed that

$$J(\boldsymbol{x}_{ij},(\alpha_i,\beta_i))=\sum_{1\le j<k\le n_i}\frac{4\left|x_{ij}-x_{ik}\right|}{\alpha_i(1+\beta_i/x_{ij})(1+\beta_i/x_{ik})}. \tag{27}$$

Therefore, by applying (24), the generalized fiducial distribution of (α_i, β_i) becomes

$$\begin{aligned} f(\alpha_i,\beta_i\mid\boldsymbol{x}_{ij}) &\propto J(\boldsymbol{x}_{ij},(\alpha_i,\beta_i))L(\boldsymbol{x}_{ij}\mid\alpha_i,\beta_i)\\ &\propto \sum_{1\le j<k\le n_i}\frac{4\left|x_{ij}-x_{ik}\right|}{\alpha_i(1+\beta_i/x_{ij})(1+\beta_i/x_{ik})}\times\\ &\frac{1}{\alpha_i^{n_i}\beta_i^{n_i}}\prod_{j=1}^{n_i}\left[\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}}+\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right]\times exp\left[-\sum_{j=1}^{n_i}\frac{1}{2\alpha_i^2}\left(\frac{x_{ij}}{\beta_i}+\frac{\beta_i}{x_{ij}}-2\right)\right] \end{aligned} \tag{28}$$

Since $J(\boldsymbol{x}_{ij},(\alpha_i,\beta_i))$ plays the same role as the prior distribution in Bayesian methodology (Li & Xu 2016), it can be applied in a similar way to the Bayesian posterior distribution to obtain the fiducial estimates of α_i and β_i (denoted as $\tilde{\alpha}_i^*$ and $\tilde{\beta}_i^*$, respectively) from the generalized fiducial distribution. Hence, the *arms* function in package *dlm* of R software was applied to obtain $\tilde{\alpha}_i^*$ and $\tilde{\beta}_i^*$. According to Equation (10), the fiducial estimates of ω become

$$\tilde{\omega}=\frac{\tilde{\alpha}_1^*\sqrt{1+\frac{5}{4}(\tilde{\alpha}_1^*)^2}\bigg/\left(1+\frac{1}{2}(\tilde{\alpha}_1^*)^2\right)}{\tilde{\alpha}_2^*\sqrt{1+\frac{5}{4}(\tilde{\alpha}_2^*)^2}\bigg/\left(1+\frac{1}{2}(\tilde{\alpha}_2^*)^2\right)}. \quad (29)$$

Based on FGCI, the 100(1-γ)% confidence interval for ω is

$$CI_{FGCI}=[L_\omega,U_\omega]=[\tilde{\omega}(\gamma/2),\tilde{\omega}(1-\gamma/2)], \quad (30)$$

where $\tilde{\omega}(\nu)$ is the 100% percentile of $\tilde{\omega}$. FGCI can be obtained by using the following algorithm.

The algorithm for FGCI

1. Generate $\boldsymbol{x}_{ij}$, $i=1,2$, $j=1,2,...,n_i$ from a BS distribution. 2. Generate K samples of α_i and β_i by using the *arms* function. 3. Burn-in P samples and keep the remaining (K - P) samples. 4. Since the generated sample is dependent, one way to reduce autocorrelation is to thin the sample. Therefore, select sampling lag $L > 1$, which retains the final number of samples as $K' = (K - P) / L$. 5. Compute the fiducial estimates of ω and obtain $\tilde{\omega}_{(1)},\tilde{\omega}_{(2)},...,\tilde{\omega}_{(K')}$. 6. Compute $\tilde{\omega}(\gamma/2)$ and $\tilde{\omega}(1-\gamma/2)$.

BAYESIAN CREDIBLE INTERVAL (BAYCI)

Bayesian inference combines the observed data (through the likelihood function) with prior information about a parameter (through the prior distribution) and expresses the combination of these in terms of the posterior distribution of the parameter based on Bayes' theorem (Bayes 1763). Since the independent Jeffreys' prior of the BS distribution results in an improper posterior distribution and the continuous conjugate joint prior distribution does not exist, Wang et al. (2016) proposed an efficient sampling algorithm based on the generalized ratio-of-uniforms method to generate samples from the posterior distribution.

Let $\boldsymbol{x}_{ij}=(x_{i1},x_{i2},...,x_{in_i}), i=1,2, j=1,2,...,n_i$ be a sample from $BS(\alpha_i,\beta_i)$, then the likelihood function is given by (26). Proper priors with known hyperparameters are considered to guarantee the propriety of the subsequent posteriors. It is assumed that $\beta_i \sim IG(a_{i,1},b_{i,1})$ and $\alpha_i^2 \sim IG(a_{i,2},b_{i,2})$, where $i=1,2$ and $IG(a,b)$ refers to an inverse gamma (IG) distribution with parameters a and b. Hence, if $Z\sim IG(a,b)$, the pdf of the IG distribution is given by

$$\pi(z,a,b)=\frac{b^a}{\Gamma(a)}z^{-a-1}\exp\left(-\frac{b}{z}\right), \quad a,b>0 \quad (31)$$

Subsequently, the joint posterior density of α_i^2 and β_i can be expressed as

$$p(\alpha_i^2,\beta_i\mid\boldsymbol{x}_{ij})\propto L(\boldsymbol{x}_{ij}\mid\alpha_i,\beta_i)\pi(\beta_i\mid a_{i,1},b_{i,1})\pi(\alpha_i^2\mid a_{i,2},b_{i,2})$$

$$\propto\frac{1}{(\alpha_i^2)^{\frac{n_i}{2}}\beta_i^{n_i}}\prod_{j=1}^{n_i}\left[\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}}+\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right]exp\left[-\sum_{j=1}^{n_i}\frac{1}{2\alpha_i^2}\left(\frac{x_{ij}}{\beta_i}+\frac{\beta_i}{x_{ij}}-2\right)\right] \quad (32)$$

$$\times\beta_i^{-a_{i,1}-1}exp\left(-\frac{b_{i,1}}{\beta_i}\right)(\alpha_i^2)^{-a_{i,2}-1}exp\left(-\frac{b_{i,2}}{\alpha_i^2}\right)$$

Hence, it follows that the marginal posterior distribution of β_i takes the form

$$\pi(\beta_i,\boldsymbol{x}_{ij})\propto\beta_i^{-(n_i+a_{i,1}+1)}exp\left(-\frac{b_{i,1}}{\beta_i}\right)\prod_{j=1}^{n_i}\left[\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}}+\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right]$$

$$\times\left[-\sum_{j=1}^{n_i}\frac{1}{2\alpha_i^2}\left(\frac{x_{ij}}{\beta_i}+\frac{\beta_i}{x_{ij}}-2\right)+b_{i,2}\right]^{-(n_i+1)/2-a_{i,2}} \quad (33)$$

By applying Equation (32), the conditional posterior distribution of α_i^2 given β_i and the data becomes

$$\pi(\alpha_i^2\mid\boldsymbol{x}_{ij},\beta_i)\propto IG\left(\frac{n_i}{2}+a_{i,2},\frac{1}{2}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{\beta_i}+\frac{\beta_i}{x_{ij}}-2\right)+b_{i,2}\right). \quad (34)$$

Markov Chain Monte Carlo methods can be used to generate samples from a posterior distribution. At the mth iteration, for m = 1, 2, ..., M, a new value, $\beta_{i(m)}$, is obtained by adopting the generalized ratio-of-uniforms method. Next, $\alpha_{i,(m)}^2$ is generated from the IG distribution given in (34) depending on $\beta_{i(m)}$. Following this, a new value, $\alpha_{i,(m)}$, is the square root of $\alpha_{i,(m)}^2$. Note that we used the R-package of *LearnBayes* to generate $\alpha^2_{i,(m)}$. The generalized ratio-of-uniforms method is explained in the following subsection.

THE GENERALIZED RATIO-OF-UNIFORMS METHOD

The generalized ratio-of-uniforms method (Wakefield et al. 1991) was used to generate β_i, for i = 1, 2, with the marginal posterior distribution in (33). Variables (u_i, v_i) are uniformly distributed in

$$C(r_i) = \left\{ (u_i, v_i) : 0 < u_i \le \left[\pi\left(\frac{v_i}{u_i^{r_i}} \mid \boldsymbol{x}_{ij} \right) \right]^{1/(r_i+1)} \right\}, \quad (35)$$

where $r_i \ge 0$ is a constant and $\pi\left(\cdot \mid \boldsymbol{x}_{ij}\right)$ is from (33). Consequently, $\beta_i = v_i \big/ u_i^{r_i}$ has a density function in the form $\pi(\beta_i \mid \boldsymbol{x}_{ij}) \big/ \int \pi(\beta_i \mid \boldsymbol{x}_{ij}) d\beta_i$.

In general, it is not possible to generate (u_i, v_i) uniformly over $C(r_i)$ directly. Hence, the accept-reject method from a convenient one-dimensional enveloping rectangle $\left[0, a(r_i)\right] \times \left[b^-(r_i), b^+(r_i)\right]$ is applied to generate random samples uniformly distributed in $C(r_i)$, where $a(r_i) = \sup_{\beta_i > 0} \left\{ \left[\pi(\beta_i \mid \boldsymbol{x}_{ij}) \right]^{1/(r_i+1)} \right\}$, $b^-(r_i) = \inf_{\beta_i > 0} \left\{ \beta_i \left[\pi(\beta_i \mid \boldsymbol{x}_{ij}) \right]^{r_i/(r_i+1)} \right\}$ and $b^+(r_i) = \sup_{\beta_i > 0} \left\{ \beta_i \left[\pi(\beta_i \mid \boldsymbol{x}_{ij}) \right]^{r_i/(r_i+1)} \right\}$. Wang et al. (2016) showed that $\pi(\beta_i \mid \boldsymbol{x}_{ij}) \to 0$ as $\beta_i \to 0^+$ and that $\pi(\beta_i \mid \boldsymbol{x}_{ij}) \to O(\beta_i^{-(a_{i,1}+a_{i,2}+3/2)})$ as $\beta_i \to +\infty$. Thus, $b^-(r_i) = 0$ and $a(r_i)$ is finite. Moreover, $b^+(r_i)$ is also finite when an appropriate r_i is chosen such that $(a_{i,1} + a_{i,2} + 3/2) r_i / (r_i + 1) - 1 > 0$ (i.e., $r > 1/(a_{i,1} + a_{i,2} + 1/2)$). Therefore, the procedure for computing BayCI can be summarized in the following steps. 1. Set the values for $a_{i,1}$, $b_{i,1}$, $a_{i,2}$, $b_{i,2}$ and r_i, for $i = 1, 2$. 2. Compute the corresponding values of $a(r_i)$ and $b^+(r_i)$. 3. At the mth step, generate $u_i \sim Unif(0, a(r_i))$ and $v_i \sim Unif(0, b^+(r_i))$, where $Unif(c, d)$ is a uniform distribution with parameters c and d, and compute $\rho_i = v_i \big/ u_i^{r_i}$. 4. If $u_i \le \left[\pi(\rho_i \mid \boldsymbol{x}_{ij}) \right]^{1/(r_i+1)}$ accept ρ_i and set $\beta_{i,(m)} = \rho_i$; otherwise, reject , and regenerate $u_i \sim Unif(0, a(r_i))$ and $v_i \sim Unif(0, b^+(r_i))$, and compute ρ_i. 5. Generate $\alpha^2_{i,(m)} \sim IG\left(\frac{n_i}{2} + a_{i,2}, \frac{1}{2} \sum_{j=1}^{n_i} \left(\frac{x_{ij}}{\beta_i} + \frac{\beta_i}{x_{ij}} - 2 \right) + b_{i,2} \right)$ and obtain $\alpha_{i,(m)}$ after a simple algebraic transformation. 6. Compute the estimator of ω by applying

$$\hat{\omega}^*_{(m)} = \frac{\alpha_{1,(m)} \sqrt{1 + \frac{5}{4}\alpha^2_{1,(m)}} \Big/ \left(1 + \frac{1}{2}\alpha^2_{1,(m)}\right)}{\alpha_{2,(m)} \sqrt{1 + \frac{5}{4}\alpha^2_{2,(m)}} \Big/ \left(1 + \frac{1}{2}\alpha^2_{2,(m)}\right)}, \quad (36)$$

7. Repeat Steps (3) to (6), M times. 8. Calculate the 100(1-γ)% credible interval by applying

$$CI_{BayCI} = \left[L_\omega, U_\omega\right] = \left[\hat{\omega}^*(\gamma/2), \hat{\omega}^*(1-\gamma/2)\right], \quad (37)$$

where $\hat{\omega}^*(\nu)$ is the 100% percentile of $\hat{\omega}^*$.

THE HIGHEST POSTERIOR DENSITY (HPD) INTERVAL

The HPD interval has two important properties: (i) the densities of the points inside the interval are higher than those of the point outside the interval; and (ii) it provides the narrowest length of the interval containing 100(1-γ)% of the posterior probability (Box & Tiao 1992). Therefore, at steps (8) in the previous section, we calculated the HPD interval by applying the package *HDInterval* version 0.2.2 from R version 3.5.1 in the simulations and computations.

SIMULATION STUDIES

The performances of the five methods derived in the previous section were evaluated in terms of their coverage probabilities and average lengths through a Monte Carlo simulation study using R statistical software. The nominal confidence level was set at 1-γ = 0.95. In comparison, a confidence interval becomes the best choice for a particular scenario when its coverage probability is above or close to 0.95 and its average length is the shortest. For the parameter configurations, the sample sizes (n_1, n_2) were set at (10,10), (20, 20), (30,30), (50,50) and (100,100) for equal sample sizes and (10, 20), (30, 20), (30,50) and (100,50) for unequal sample sizes. Since β is the scale parameter, $\beta_1 = \beta_2 = 1$were fixed to avoid loss of generality, while the different values of shape parameters (α_1, α_2) were considered: (0.25, 0.25), (0.25, 0.50), (0.25, 1.00), (0.25, 2.00), (0.25, 3.00), (0.50, 0.50), (0.50, 1.00), (0.50, 2.00), (0.50, 3.00), (1.00, 1.00), (1.00, 2.00), (1.00, 3.00), (2.00, 2.00), (2.00, 3.00) and (3.00, 3.00). For each combination of sample sizes (n_1, n_2) and the shape parameters (α_1, α_2), the simulation results were obtained after running 1,000 replications, with 5,000 pivotal quantities for GCI, B = 500 for BCI,

$K = 3{,}000$ for FGCI and $M = 1{,}000$ for BayCI and the HPD interval. Wang et al. (2016) showed that the Bayesian estimation of the BS distribution is insensitive to the choice of r_i and the hyperparameters $a_{i,1}, b_{i,1}$, $a_{i,2}$ and $b_{i,2}$, for $i = 1, 2$. They recommended that r_i should be an integer that is $r_i \geq 1$ and the hyperparameters should be set such that $a_{i,1} = b_{i,1} = a_{i,2} = b_{i,2} \leq 10^{-3}$. Moreover, Congdon (2001) suggested that the hyperparameters should be set close to 0. Therefore, $r_1 = r_2 = 2$ and $a_{i,1} = b_{i,1} = a_{i,2} = b_{i,2} \leq 10^{-4}$ were used for BayCI. The simulation results for equal and unequal sample sizes are reported in Tables 1 and 2, respectively. Figures 1 and 2 summarize the coverage probabilities and average lengths of the methods, respectively.

TABLE 1. The coverage probabilities and average lengths of five methods for the nominal 95% confidence intervals for the ratio of the CVs of BS distributions with equal sample sizes ($n_1 = n_2$)

(n_1, n_2)	(α_1, α_2)	Coverage probability (Average length)				
		GCI	BCI	FGCI	BayCI	HPD
(10,10)	(0.25,0.25)	0.962	0.913	0.946	0.963	0.952
		(1.6366)	(1.3267)	(1.5187)	(1.6090)	(1.5066)
	(0.25,0.50)	0.957	0.898	0.937	0.952	0.942
		(0.7993)	(0.6460)	(0.7424)	(0.7863)	(0.7333)
	(0.25,1.00)	0.953	0.899	0.942	0.952	0.953
		(0.3958)	(0.3192)	(0.3694)	(0.3895)	(0.3606)
	(0.25,2.00)	0.938	0.877	0.924	0.936	0.945
		(0.2073)	(0.1609)	(0.1904)	(0.2035)	(0.1880)
	(0.25,3.00)	0.946	0.890	0.926	0.938	0.942
		(0.1636)	(0.1227)	(0.1483)	(0.1611)	(0.1484)
	(0.50,0.50)	0.951	0.909	0.941	0.950	0.946
		(1.5860)	(1.2971)	(1.4869)	(1.5589)	(1.4605)
	(0.50,1.00)	0.950	0.892	0.934	0.944	0.948
		(0.7809)	(0.6377)	(0.7322)	(0.7667)	(0.7141)
	(0.50,2.00)	0.944	0.889	0.933	0.940	0.946
		(0.4186)	(0.3300)	(0.3884)	(0.4120)	(0.3827)
	(0.50,3.00)	0.948	0.883	0.939	0.942	0.943
		(0.3232)	(0.2455)	(0.2951)	(0.3169)	(0.2952)
	(1.00,1.00)	0.956	0.908	0.941	0.949	0.949
		(1.3154)	(1.1447)	(1.2597)	(1.2921)	(1.2311)
	(1.00,2.00)	0.952	0.898	0.942	0.951	0.945
		(0.6528)	(0.5518)	(0.6204)	(0.6380)	(0.6105)
	(1.00,3.00)	0.936	0.872	0.925	0.934	0.930
		(0.5003)	(0.4101)	(0.4702)	(0.4898)	(0.4703)
	(2.00,2.00)	0.953	0.882	0.936	0.949	0.947
		(0.7591)	(0.7096)	(0.7437)	(0.7377)	(0.7188)
	2.00,3.00)	0.956	0.885	0.934	0.948	0.950
		(0.5205)	(0.4788)	(0.5063)	(0.5041)	(0.4956)
	(3.00,3.00)	0.959	0.887	0.938	0.947	0.950
		(0.4640)	(0.4550)	(0.4600)	(0.4494)	(0.4417)

(20,20)	(0.25,0.25)	0.951	0.932	0.946	0.949	0.953
		(1.0252)	(0.9296)	(0.9939)	(0.0172)	(0.9824)
	(0.25,0.50)	0.947	0.925	0.937	0.948	0.945
		(0.4959)	(0.4504)	(0.4810)	(0.4926)	(0.4750)
	(0.25,1.00)	0.914	0.921	0.931	0.942	0.942
		(0.2380)	(0.2153)	(0.2306)	(0.2365)	(0.2272)
	(0.25,2.00)	0.957	0.926	0.943	0.949	0.947
		(0.1245)	(0.1105)	(0.1197)	(0.1239)	(0.1186)
	(0.25,3.00)	0.962	0.923	0.948	0.959	0.948
		(0.0962)	(0.0836)	(0.0918)	(0.0959)	(0.0917)
	(0.50,0.50)	0.942	0.918	0.937	0.945	0.937
		(0.9951)	(0.9073)	(0.9637)	(0.9867)	(0.9547)
	(0.50,1.00)	0.954	0.929	0.947	0.950	0.949
		(0.4821)	(0.4377)	(0.4673)	(0.4770)	(0.4594)
	(0.50,2.00)	0.948	0.925	0.938	0.946	0.943
		(0.2489)	(0.2213)	(0.2400)	(0.2472)	(0.2369)
	(0.50,3.00)	0.965	0.931	0.958	0.961	0.964
		(0.1959)	(0.1707)	(0.1873)	(0.1945)	(0.1869)
	(1.00,1.00)	0.948	0.923	0.943	0.947	0.940
		(0.8383)	(0.7823)	(0.8201)	(0.8303)	(0.8073)
	(1.00,2.00)	0.956	0.923	0.949	0.953	0.948
		(0.4193)	(0.3858)	(0.4075)	(0.4149)	(0.4036)
	(1.00,3.00)	0.946	0.928	0.943	0.936	0.944
		(0.3244)	(0.2925)	(0.3142)	(0.3205)	(0.3133)
	(2.00,2.00)	0.964	0.927	0.951	0.957	0.951
		(0.4836)	(0.4748)	(0.4757)	(0.4736)	(0.4660)
	2.00,3.00)	0.959	0.922	0.947	0.953	0.953
		(0.3355)	(0.3235)	(0.3280)	(0.3265)	(0.3230)
	(3.00,3.00)	0.942	0.913	0.933	0.940	0.945
		(0.2893)	(0.2875)	(0.2831)	(0.2785)	(0.2753)
(30,30)	(0.25,0.25)	0.950	0.937	0.946	0.947	0.953
		(0.7871)	(0.7376)	(0.7700)	(0.7838)	(0.7637)
	(0.25,0.50)	0.958	0.945	0.958	0.957	0.958
		(0.3878)	(0.3638)	(0.3793)	(0.3867)	(0.3763)
	(0.25,1.00)	0.940	0.923	0.934	0.938	0.933
		(0.1875)	(0.1753)	(0.1839)	(0.1870)	(0.1816)
	(0.25,2.00)	0.953	0.938	0.955	0.953	0.952
		(0.0963)	(0.0889)	(0.0940)	(0.0959)	(0.0929)
	(0.25,3.00)	0.954	0.939	0.951	0.951	0.952
		(0.0763)	(0.0693)	(0.0740)	(0.0759)	(0.0736)

	(0.50,0.50)	0.950	0.932	0.949	0.950	0.951
		(0.7855)	(0.7377)	(0.7682)	(0.7811)	(0.7616)
	(0.50,1.00)	0.948	0.937	0.945	0.947	0.947
		(0.3742)	(0.3502)	(0.3660)	(0.3713)	(0.3612)
	(0.50,2.00)	0.953	0.941	0.946	0.952	0.956
		(0.1960)	(0.1814)	(0.1909)	(0.1952)	(0.1894)
	(0.50,3.00)	0.947	0.929	0.944	0.942	0.940
		(0.1527)	(0.1388)	(0.1483)	(0.1523)	(0.1478)
	(1.00,1.00)	0.955	0.933	0.949	0.955	0.951
		(0.6650)	(0.6358)	(0.6544)	(0.6606)	(0.6465)
	(1.00,2.00)	0.945	0.928	0.942	0.941	0.941
		(0.3357)	(0.3170)	(0.3297)	(0.3328)	(0.3262)
	(1.00,3.00)	0.944	0.929	0.943	0.947	0.942
		(0.2585)	(0.2399)	(0.2524)	(0.2565)	(0.2516)
	(2.00,2.00)	0.958	0.927	0.942	0.945	0.947
		(0.3830)	(0.3871)	(0.3775)	(0.3769)	(0.3718)
	2.00,3.00)	0.960	0.939	0.950	0.959	0.960
		(0.2644)	(0.2581)	(0.2595)	(0.2592)	(0.2568)
	(3.00,3.00)	0.959	0.941	0.946	0.953	0.952
		(0.2236)	(0.2233)	(0.2189)	(0.2173)	(0.2152)
(50,50)	(0.25,0.25)	0.952	0.948	0.948	0.952	0.949
		(0.5895)	(0.5669)	(0.5809)	(0.5864)	(0.5760)
	(0.25,0.50)	0.956	0.949	0.952	0.956	0.955
		(0.2917)	(0.2791)	(0.2883)	(0.2901)	(0.2849)
	(0.25,1.00)	0.957	0.944	0.953	0.957	0.954
		(0.1402)	(0.1341)	(0.1382)	(0.1394)	(0.1366)
	(0.25,2.00)	0.943	0.941	0.942	0.942	0.947
		(0.0725)	(0.0690)	(0.0713)	(0.0723)	(0.0708)
	(0.25,3.00)	0.945	0.926	0.943	0.946	0.942
		(0.0568)	(0.0534)	(0.0556)	(0.0556)	(0.0553)
	(0.50,0.50)	0.934	0.923	0.924	0.935	0.932
		(0.5880)	(0.5648)	(0.5792)	(0.5862)	(0.5750)
	(0.50,1.00)	0.946	0.935	0.945	0.947	0.944
		(0.2831)	(0.2721)	(0.2793)	(0.2818)	(0.2761)
	(0.50,2.00)	0.952	0.943	0.951	0.951	0.952
		(0.1452)	(0.1378)	(0.1427)	(0.1448)	(0.1416)
	(0.50,3.00)	0.948	0.928	0.939	0.939	0.939
		(0.1151)	(0.1081)	(0.1127)	(0.1146)	(0.1122)
	(1.00,1.00)	0.951	0.934	0.944	0.951	0.946
		(0.5053)	(0.4903)	(0.4995)	(0.5019)	(0.4943)

	(1.00,2.00)	0.952	0.944	0.953	0.954	0.950
		(0.2526)	(0.2439)	(0.2499)	(0.2513)	(0.2474)
	(1.00,3.00)	0.951	0.930	0.943	0.953	0.948
		(0.1958)	(0.1869)	(0.1927)	(0.1946)	(0.1917)
	(2.00,2.00)	0.941	0.929	0.936	0.943	0.938
		(0.2885)	(0.2866)	(0.2859)	(0.2860)	(0.2829)
	2.00,3.00)	0.959	0.948	0.954	0.953	0.955
		(0.1998)	(0.1959)	(0.1976)	(0.1965)	(0.1947)
	(3.00,3.00)	0.949	0.940	0.943	0.940	0.941
		(0.1662)	(0.1654)	(0.1638)	(0.1629)	(0.1614)
(100,100)	(0.25,0.25)	0.950	0.936	0.947	0.950	0.944
		(0.4062)	(0.3954)	(0.4026)	(0.4045)	(0.3991)
	(0.25,0.50)	0.945	0.941	0.947	0.945	0.939
		(0.2007)	(0.1955)	(0.1985)	(0.1997)	(0.1969)
	(0.25,1.00)	0.936	0.938	0.941	0.940	0.937
		(0.0953)	(0.0931)	(0.0946)	(0.0950)	(0.0936)
	(0.25,2.00)	0.952	0.951	0.952	0.951	0.952
		(0.0494)	(0.0479)	(0.0488)	(0.0492)	(0.0485)
	(0.25,3.00)	0.954	0.936	0.949	0.951	0.946
		(0.0391)	(0.0378)	(0.0387)	(0.0390)	(0.0384)
	(0.50,0.50)	0.943	0.933	0.942	0.940	0.936
		(0.4059)	(0.3971)	(0.4028)	(0.4051)	(0.3998)
	(0.50,1.00)	0.960	0.948	0.957	0.956	0.953
		(0.1955)	(0.1910)	(0.1936)	(0.1948)	(0.1920)
	(0.50,2.00)	0.950	0.937	0.948	0.948	0.948
		(0.0999)	(0.0966)	(0.0991)	(0.0995)	(0.0980)
	(0.50,3.00)	0.932	0.922	0.929	0.928	0.930
		(0.0796)	(0.0770)	(0.0787)	(0.0794)	(0.0782)
	(1.00,1.00)	0.952	0.946	0.951	0.954	0.944
		(0.3541)	(0.3479)	(0.3518)	(0.3530)	(0.3487)
	(1.00,2.00)	0.947	0.945	0.948	0.944	0.945
		(0.1753)	(0.1713)	(0.1738)	(0.1745)	(0.1724)
	(1.00,3.00)	0.938	0.935	0.937	0.932	0.926
		(0.1368)	(0.1331)	(0.1353)	(0.1361)	(0.1345)
	(2.00,2.00)	0.952	0.942	0.949	0.957	0.950
		(0.1992)	(0.1976)	(0.1981)	(0.1978)	(0.1959)
	2.00,3.00)	0.950	0.942	0.948	0.947	0.945
		(0.1382)	(0.1364)	(0.1374)	(0.1373)	(0.1361)
	(3.00,3.00)	0.950	0.937	0.941	0.949	0.948
		(0.1142)	(0.1136)	(0.1130)	(0.1128)	(0.1118)

TABLE 2. The coverage probabilities and average lengths of five methods for the nominal 95% confidence intervals for the ratio of the CVs of BS distributions with unequal sample sizes ($n_1 \neq n_2$)

(n_1, n_2)	(α_1, α_2)	Coverage probability (Average length)				
		GCI	BCI	FGCI	BayCI	HPD
(10,20)	(0.25,0.25)	0.959	0.908	0.947	0.961	0.952
		(1.3624)	(1.0617)	(1.2382)	(1.3345)	(1.2419)
	(0.25,0.50)	0.948	0.894	0.928	0.943	0.940
		(0.6874)	(0.5353)	(0.6248)	(0.6719)	(0.6256)
	(0.25,1.00)	0.957	0.908	0.945	0.947	0.951
		(0.3389)	(0.2618)	(0.3074)	(0.3318)	(0.3072)
	(0.25,2.00)	0.939	0.903	0.935	0.933	0.945
		(0.1911)	(0.1419)	(0.1708)	(0.1874)	(0.1729)
	(0.25,3.00)	0.939	0.879	0.923	0.931	0.941
		(0.1529)	(0.1107)	(0.1357)	(0.1498)	(0.1380)
	(0.50,0.50)	0.946	0.895	0.940	0.945	0.945
		(1.3652)	(1.0797)	(1.2520)	(1.3403)	(1.2565)
	(0.50,1.00)	0.948	0.902	0.933	0.942	0.944
		(0.6698)	(0.5269)	(0.6112)	(0.6551)	(0.6123)
	(0.50,2.00)	0.946	0.897	0.939	0.943	0.952
		(0.3695)	(0.2784)	(0.3344)	(0.3628)	(0.3386)
	(0.50,3.00)	0.960	0.906	0.951	0.948	0.956
		(0.3031)	(0.2234)	(0.2720)	(0.2970)	(0.2770)
	(1.00,1.00)	0.945	0.892	0.935	0.935	0.936
		(1.0766)	(0.9136)	(1.0116)	(1.0525)	(1.0111)
	(1.00,2.00)	0.943	0.878	0.932	0.942	0.925
		(0.5765)	(0.4707)	(0.5359)	(0.5631)	(0.5414)
	(1.00,3.00)	0.933	0.876	0.929	0.929	0.927
		(0.4640)	(0.3685)	(0.4295)	(0.4524)	(0.4367)
	(2.00,2.00)	0.958	0.884	0.930	0.954	0.938
		(0.5996)	(0.5506)	(0.5764)	(0.5801)	(0.5710)
	2.00,3.00)	0.958	0.882	0.944	0.953	0.938
		(0.4522)	(0.4013)	(0.4321)	(0.4359)	(0.4307)
	(3.00,3.00)	0.956	0.899	0.938	0.945	0.950
		(0.3637)	(0.3507)	(0.3548)	(0.3500)	(0.3465)
(30,20)	(0.25,0.25)	0.947	0.931	0.943	0.945	0.943
		(0.8943)	(0.8431)	(0.8775)	(0.8900)	(0.8683)
	(0.25,0.50)	0.960	0.941	0.959	0.960	0.959
		(0.4422)	(0.4166)	(0.4338)	(0.4396)	(0.4282)
	(0.25,1.00)	0.947	0.920	0.935	0.944	0.942
		(0.2099)	(0.1986)	(0.2069)	(0.2088)	(0.2029)

	(0.25,2.00)	0.948	0.930	0.943	0.948	0.947
		(0.1026)	(0.0958)	(0.1004)	(0.1021)	(0.0989)
	(0.25,3.00)	0.952	0.936	0.944	0.949	0.945
		(0.0789)	(0.0725)	(0.0768)	(0.0787)	(0.0761)
	(0.50,0.50)	0.943	0.910	0.935	0.943	0.944
		(0.8914)	(0.8405)	(0.8763)	(0.8879)	(0.8655)
	(0.50,1.00)	0.946	0.921	0.940	0.946	0.949
		(0.4258)	(0.4043)	(0.4194)	(0.4241)	(0.4119)
	(0.50,2.00)	0.951	0.934	0.949	0.950	0.951
		(0.2085)	(0.1948)	(0.2044)	(0.2072)	(0.2009)
	(0.50,3.00)	0.954	0.939	0.949	0.948	0.950
		(0.1601)	(0.1471)	(0.1559)	(0.1592)	(0.1544)
	(1.00,1.00)	0.941	0.911	0.928	0.939	0.932
		(0.7640)	(0.7344)	(0.7544)	(0.7566)	(0.7401)
	(1.00,2.00)	0.945	0.927	0.944	0.949	0.940
		(0.3620)	(0.3466)	(0.3573)	(0.3594)	(0.3509)
	(1.00,3.00)	0.958	0.932	0.943	0.942	0.940
		(0.2698)	(0.2539)	(0.2643)	(0.2674)	(0.2625)
	(2.00,2.00)	0.950	0.917	0.939	0.946	0.944
		(0.4429)	(0.4390)	(0.4391)	(0.4352)	(0.4284)
	2.00,3.00)	0.965	0.937	0.955	0.958	0.959
		(0.2882)	(0.2834)	(0.2839)	(0.2822)	(0.2789)
	(3.00,3.00)	0.951	0.916	0.942	0.944	0.947
		(0.2604)	(0.2614)	(0.2562)	(0.2526)	(0.2491)
(30,50)	(0.25,0.25)	0.950	0.932	0.941	0.941	0.940
		(0.6905)	(0.6401)	(0.6726)	(0.6870)	(0.6682)
	(0.25,0.50)	0.949	0.934	0.944	0.948	0.946
		(0.3457)	(0.3202)	(0.3372)	(0.3437)	(0.3343)
	(0.25,1.00)	0.950	0.936	0.946	0.951	0.945
		(0.1660)	(0.1534)	(0.1616)	(0.1652)	(0.1605)
	(0.25,2.00)	0.951	0.938	0.949	0.945	0.954
		(0.0917)	(0.0835)	(0.0890)	(0.0913)	(0.0885)
	(0.25,3.00)	0.954	0.933	0.949	0.953	0.947
		(0.0743)	(0.0669)	(0.0718)	(0.0741)	(0.0716)
	(0.50,0.50)	0.939	0.924	0.933	0.939	0.930
		(0.6989)	(0.6470)	(0.6822)	(0.6945)	(0.6769)
	(0.50,1.00)	0.943	0.933	0.938	0.942	0.937
		(0.3346)	(0.3096)	(0.3257)	(0.3335)	(0.3242)
	(0.50,2.00)	0.947	0.937	0.947	0.945	0.949
		(0.1839)	(0.1679)	(0.1784)	(0.1830)	(0.1778)
	(0.50,3.00)	0.956	0.927	0.949	0.953	0.946
		(0.1496)	(0.1355)	(0.1450)	(0.1488)	(0.1445)

	(1.00,1.00)	0.946	0.926	0.944	0.946	0.943
		(0.5930)	(0.5615)	(0.5817)	(0.5891)	(0.5774)
	(1.00,2.00)	0.949	0.920	0.946	0.948	0.941
		(0.3128)	(0.2917)	(0.3052)	(0.3100)	(0.3044)
	(1.00,3.00)	0.951	0.933	0.948	0.955	0.945
		(0.2516)	(0.2318)	(0.2444)	(0.2493)	(0.2446)
	(2.00,2.00)	0.944	0.919	0.934	0.946	0.937
		(0.3349)	(0.3261)	(0.3290)	(0.3297)	(0.3262)
	2.00,3.00)	0.950	0.931	0.945	0.946	0.942
		(0.2466)	(0.2372)	(0.2416)	(0.2419)	(0.2398)
	(3.00,3.00)	0.958	0.940	0.948	0.948	0.953
		(0.1955)	(0.1927)	(0.1914)	(0.1900)	(0.1884)
(100,50)	(0.25,0.25)	0.945	0.933	0.936	0.941	0.940
		(0.4983)	(0.4873)	(0.4937)	(0.4958)	(0.4898)
	(0.25,0.50)	0.934	0.932	0.935	0.936	0.929
		(0.2460)	(0.2405)	(0.2446)	(0.2450)	(0.2418)
	(0.25,1.00)	0.949	0.934	0.948	0.949	0.944
		(0.1173)	(0.1152)	(0.1168)	(0.1168)	(0.1151)
	(0.25,2.00)	0.950	0.935	0.947	0.951	0.947
		(0.0548)	(0.0537)	(0.0544)	(0.0546)	(0.0537)
	(0.25,3.00)	0.951	0.945	0.947	0.951	0.943
		(0.0410)	(0.0399)	(0.0407)	(0.0409)	(0.0403)
	(0.50,0.50)	0.944	0.929	0.935	0.941	0.943
		(0.4999)	(0.4909)	(0.4972)	(0.4987)	(0.4923)
	(0.50,1.00)	0.948	0.931	0.944	0.944	0.939
		(0.2355)	(0.2311)	(0.2339)	(0.2344)	(0.2311)
	(0.50,2.00)	0.939	0.928	0.935	0.939	0.938
		(0.1120)	(0.1098)	(0.1113)	(0.1113)	(0.1097)
	(0.50,3.00)	0.956	0.944	0.951	0.955	0.951
		(0.0826)	(0.0802)	(0.0819)	(0.0823)	(0.0811)
	(1.00,1.00)	0.953	0.943	0.949	0.953	0.952
		(0.4358)	(0.4295)	(0.4336)	(0.4335)	(0.4278)
	(1.00,2.00)	0.954	0.945	0.949	0.955	0.952
		(0.1988)	(0.1959)	(0.1979)	(0.1978)	(0.1951)
	(1.00,3.00)	0.940	0.931	0.937	0.938	0.933
		(0.1444)	(0.1410)	(0.1433)	(0.1434)	(0.1417)
	(2.00,2.00)	0.949	0.940	0.949	0.948	0.945
		(0.2484)	(0.2478)	(0.2472)	(0.2457)	(0.2427)
	2.00,3.00)	0.957	0.949	0.951	0.953	0.947
		(0.1573)	(0.1566)	(0.1562)	(0.1557)	(0.1542)
	(3.00,3.00)	0.956	0.939	0.952	0.953	0.950
		(0.1431)	(0.1433)	(0.1419)	(0.1409)	(0.1394)

The results for all scenarios of equal or unequal sample sizes were similar, and thus we can draw the following conclusions. When the sample sizes (n_1, n_2) were small (e.g., (10,10)), BCI performed poorly since its coverage probabilities were much lower than the nominal level, and although its coverage probabilities were better when sample sizes (n_1, n_2) were increased, they were still lower than the nominal level. However, BCI obtained the shortest average lengths for all cases, albeit not by much. For all scenarios, the coverage probabilities of GCI, FGCI, BayCI, and the HPD interval were above or close to the nominal level and each other. However, the HPD interval outperformed the others as its average lengths were the shortest in most cases (except when (n_1, n_2)= (10,20)) whereas those of GCI were the largest. In addition, when sample sizes (n_1, n_2) were increased, the average lengths of the five methods tended to decrease, and there was very little difference between them when sample sizes (n_1, n_2) were greater than 30.

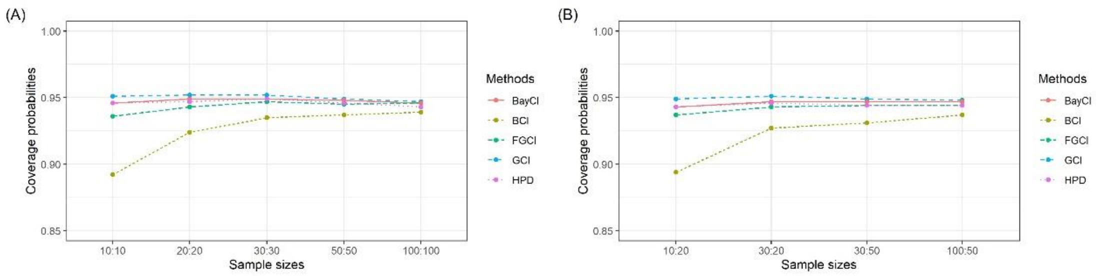

FIGURE 1. Coverage probabilities of the methods: (A) equal sample sizes and (B) unequal sample sizes

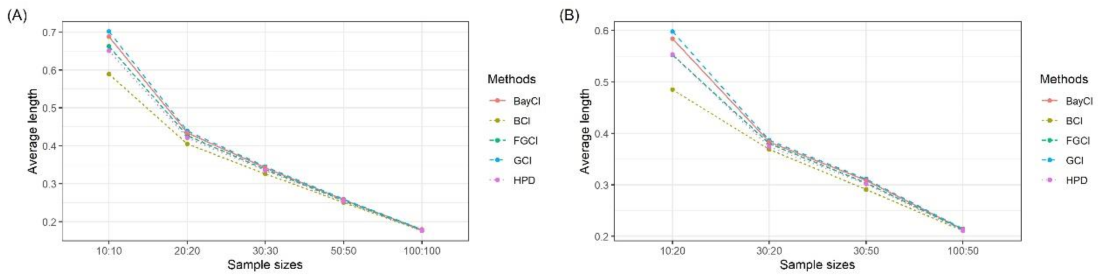

FIGURE 2. Average lengths of the methods: (A) equal sample sizes and (B) unequal sample sizes

AN EMPIRICAL APPLICATION

The proposed methods and GCI were applied to real fatigue life datasets taken from Birnbaum and Saunders (1996b) for 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The corresponding maximum stresses per cycle for groups 1 and 2 were 31,000 and 21,000 psi, respectively. Table 3 provides the descriptive statistics of the data, including the central tendency statistic, standard deviation, and CV. Hence, the ratio of the CVs was 0.5986. For BayCI, we chose $r = 2$ and hyperparameter values $a_{i,1} = b_{i,1} = a_{i,2} = b_{i,2} \leq 10^{-3}$ for both datasets. The 95% confidence intervals for the ratio of the CVs based on GCI, BCI, FGCI, BayCI, and the HPD interval are summarized in Table 4.

TABLE 3. Summary statistics for the fatigue lifetime data of the 6061-T6 aluminum coupons

Sample	n	Min	Median	Mean	Max.	SD	CV
Group 1	101	70	133	133.7327	212	22.3557	0.1672
Group 2	101	370	1416	1400.911	2440	391.3241	0.2793

The results indicate that the average lengths for BCI were once again the shortest, followed by the HPD interval. Recall that in the simulation study, the coverage probabilities of BCI were lower than 0.95 whereas those of GCI, FGCI, BayCI, and the HPD interval were higher than or close to 0.95. Therefore, when we consider the coverage probability and the average length simultaneously, the HPD interval can be recommended for constructing confidence intervals for the ratio of CVs of the BS distributions of these two datasets. From Table 4, it can be seen that the lower and upper confidence levels do not include 1, and so the CVs of fatigue lifetime of 6061-T6 aluminum coupons with maximum stresses per cycle 31,000 and 21,000 psi are different.

TABLE 4. The 95% confidence intervals and lengths using the five methods to construct confidence intervals for the ratio of the CVs of fatigue lifetime data of 6061-T6 aluminum coupons

Methods	Interval	Length
GCI	0.4448 - 0.6629	0.2181
BCI	0.4522 - 0.6590	0.2069
FGCI	0.4497 - 0.6661	0.2164
BayCI	0.4479 - 0.6624	0.2146
HPD	0.4428 - 0.6503	0.2075

CONCLUSIONS

In this study, BCI, FGCI, BayCI, and the HPD interval were used to construct confidence intervals for the ratio of the CVs of BS distributions. The performances of the proposed methods were compared with GCI through Monte Carlo simulations. In all of the test scenarios, the coverage probabilities of GCI, FGCI, BayCI, and the HPD interval were close to or above the nominal level whereas those of BCI were below, even though it obtained the shortest average lengths with HPD as the second-best. Therefore, when we consider the coverage probability and average length simultaneously, the HPD interval is recommended since its coverage probabilities were higher than or close to the nominal level in almost all cases and its average lengths were second-best. In addition, when sample sizes (n_1, n_2) were increased, the average lengths of GCI, FGCI, and BayCI were similar to the HPD interval, and so these three methods can be considered as alternatives under these circumstances.

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