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α-LOGARITHMICALLY CONVEX FUNCTIONS

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ABSTRACT

For $\alpha \geq 0$, we introduce the class M^α of normlised analytic α-logarithmically convex functions defined in the open unit disc D by

$$\text{Re}\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)^\alpha\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\right\} > 0.$$

For $f \in M^\alpha$, a best possible subordination theorem is obtained which implies that M^α forms a subset of the starlike functions S^*. Some extreme coefficient problems are also solved.

Keywords:α-logarithm; Convex Functions; Extreme Coefficient Problem; Fekete-Szegö

1. Introduction

Let $\mathcal{A}$ denote the class of normalised analytic functions f defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

for $z \in D = \{z : |z| < 1\}$. A great deal of attention has been given in recent years to the subset M_α of α-convex functions introduced by Miller et al. (1972) and defined as follows:

Definition 1.1. *Let $\alpha \in \mathbb{R}$. A function $f \in \mathcal{A}$ is said to be α-convex, if, for $z \in D$ and $\frac{f(z)}{z}f'(z) \neq 0$,*

$$Re\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)\right\} > 0.$$

2. Results

We begin by stating the following lemma of Miller and Mocanu (1981).

Lemma 2.1. *Let F be analytic in D and G be analytic and univalent in $\overline{D}$ except for points ζ such that $\lim_{z\to\zeta} F(z) = \infty$, with $F(0) = G(0)$. If $f \not\prec G$, then there is a point $z_0 \in D$ and $\zeta_0 \in \partial D$ such that $F(|z| < |z_0|) \subset G(D)$, $F(z_0) = G(\zeta_0)$ and $z_0F'(z_0) = m\zeta_0G'(\zeta_0)$ for $m \geq 1$.*

We first give a general result as follows:

Theorem 2.2. *Let $f \in \mathcal{A}$. Suppose $\alpha \geq 0$ and $0 \leq \alpha + \beta\delta \leq 4$. Then for $0 < \delta < 1$,*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)^\alpha\left(\frac{zf'(z)}{f(z)}\right)^\beta \prec \left(\frac{1+z}{1-z}\right)^{\lambda(\delta,\alpha,\beta)},$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, \tag{2}$$

for $z \in D$, where

$$\lambda(\delta,\alpha,\beta) = \frac{2}{\pi}\alpha \arctan\left(\tan\frac{\delta\pi}{2} + \frac{\delta}{(1-\delta)^{\frac{1-\delta}{2}}(1+\delta)^{\frac{1+\delta}{2}}\cos\frac{\delta\pi}{2}}\right) + \beta\delta. \tag{3}$$

$\lambda(\delta,\alpha,\beta)$ given by Eq. (3) is the largest number such the (2) holds.

Proof. Let $p(z) = \frac{zf'(z)}{f(z)}$, so that p is analytic and $p(0) = 1$. We need to show that

⋮

⋮

Thus Eq. (3) is exact and the proof of Theorem 2.2 is complete. □

Remark 2.3. *We note that when $\alpha = 1$ and $\beta = 0$ in Theorem 2.2 we obtain the result of Nunokawa and Thomas (1996).*

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